Global regularity for Rayleigh-Taylor unstable Muskat bubbles.

Robert Strain (University of Pennsylvania)
with Collaborators:
Francisco Gancedo (University of Seville),
Eduardo García-Juárez (University of Pennsylvania),
Neel Patel (University of Michigan).

Diffusions non standards en mécanique des fluides,
équations cinétiques et probabilités,
Friday, 14 December, 2018

Introduction to the Muskat problem

Consider the general transport equation

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

Introduction to the Muskat problem

Consider the general transport equation

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

Here ρ is an "active scalar" which is driven by the incompressible velocity u:

$$\nabla \cdot u = 0.$$

This type of system comes up in many contexts in fluid dynamics and beyond by taking a suitable choice of u.

Introduction to the Muskat problem

Consider the general transport equation

$$\rho_t + u \cdot \nabla\rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

Here ρ is an "active scalar" which is driven by the incompressible velocity u:

$$\nabla \cdot u = 0.$$

This type of system comes up in many contexts in fluid dynamics and beyond by taking a suitable choice of u.

- Vortex Patch Problems

Introduction to the Muskat problem

Consider the general transport equation

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

Here ρ is an "active scalar" which is driven by the incompressible velocity u:

$$\nabla \cdot u = 0.$$

This type of system comes up in many contexts in fluid dynamics and beyond by taking a suitable choice of u.

- Vortex Patch Problems
- Surface Quasi-geostrophic equation (SQG):

$$u \stackrel{\text{def}}{=} R^{\perp}\rho = (-R_2\rho, R_1\rho), \quad \widehat{R}_j = i\frac{\xi_j}{|\xi|}$$

Introduction to the Muskat problem

Consider the general transport equation

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

Here ρ is an "active scalar" which is driven by the incompressible velocity u:

$$\nabla \cdot u = 0.$$

This type of system comes up in many contexts in fluid dynamics and beyond by taking a suitable choice of u.

- Vortex Patch Problems
- Surface Quasi-geostrophic equation (SQG):

$$u \stackrel{\text{def}}{=} R^{\perp}\rho = (-R_2\rho, R_1\rho), \quad \widehat{R}_j = i\frac{\xi_j}{|\xi|}$$

- Muskat Problem (using Darcy's law.)

The Muskat Problem

Consider two incompressible ($\nabla \cdot u(x,t) = 0$) immiscible fluids in porous media under the assumption of no surface tension. In 3D, this scenario is modeled using the classical Darcy's law

$$\mu(x,t)u(x,t) = -\nabla p(x,t) - \rho(x,t)e_3,$$

where the velocity of the fluid u is proportional to the spatial gradient pressure ∇p and the gravity force.

The Muskat Problem

Consider two incompressible ($\nabla \cdot u(x,t) = 0$) immiscible fluids in porous media under the assumption of no surface tension. In 3D, this scenario is modeled using the classical Darcy's law

$$\mu(x,t)u(x,t) = -\nabla p(x,t) - \rho(x,t)e_3,$$

where the velocity of the fluid u is proportional to the spatial gradient pressure ∇p and the gravity force.
The densities and viscosities of each fluid are given by

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t). \end{cases}$$

For the transport equation, initial data of this form propagate this structure forward in time, where $D^i(t)$ moving domains.

The Muskat Problem

Consider two incompressible ($\nabla \cdot u(x,t) = 0$) immiscible fluids in porous media under the assumption of no surface tension. In 3D, this scenario is modeled using the classical Darcy's law

$$\mu(x,t)u(x,t) = -\nabla p(x,t) - \rho(x,t)e_3,$$

where the velocity of the fluid u is proportional to the spatial gradient pressure ∇p and the gravity force.
The densities and viscosities of each fluid are given by

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t). \end{cases}$$

For the transport equation, initial data of this form propagate this structure forward in time, where $D^i(t)$ moving domains.
Widely noted similarity to Hele-Shaw (Saffman & Taylor (1958))

The Muskat problem in 2D and 3D when $A_\mu = 0$.

The Atwood number is given by

$$A_\mu = \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1} \in [0, 1].$$

Many works assume $A_\mu = 0$, in this case the Muskat problem can takes the following form.

The Muskat problem in 2D and 3D when $A_\mu = 0$.

The Atwood number is given by

$$A_\mu = \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1} \in [0, 1].$$

Many works assume $A_\mu = 0$, in this case the Muskat problem can takes the following form. In 2D when $A_\mu = 0$:

$$f_t(\alpha, t) = \frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{\beta(\partial_\alpha f(\alpha, t) - \partial_\alpha f(\alpha - \beta, t))}{\beta^2 + (f(\alpha, t) - f(\alpha - \beta, t))^2} d\beta,$$
$$f(\alpha, 0) = f_0(\alpha), \quad \alpha \in \mathbb{R}.$$

In 3D when $A_\mu = 0$:

$$f_t(x, t) = \frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}^2} \frac{(\nabla f(x, t) - \nabla f(x - y, t)) \cdot y}{[|y|^2 + (f(x, t) - f(x - y, t))^2]^{3/2}} dy,$$
$$f(x, 0) = f_0(x), \quad x \in \mathbb{R}^2.$$

Here f defines the interface, which is the free boundary ∂D^i.

(incomplete collection of) References

Constantin P, Majda AJ, Tabak E (1994); Held I, Pierrehumbert R, Garner S, Swanson K (1995); Constantin P, Nie Q, Schorghofer N (1998); Gill AE (1982); Majda AJ, Bertozzi A (2002); Ohkitani K, Yamada M (1997); Córdoba D (1998); Córdoba D, Fefferman D (2002); Deng J, Hou TY, Li R, Yu X (2006); Chae D, Constantin P, Wu J (2012); Constantin P, Lai MC, Sharma R, Tseng YH, Wu J (2012); Rodrigo JL (2005); Gancedo F (2008); Bertozzi AL, Constantin P (1993); Fefferman C, Rodrigo JL (2011); Córdoba D, Fontelos MA, Mancho AM, Rodrigo JL (2005); Fefferman C, Rodrigo JL (2012); Otto F (1999); Córdoba D, Gancedo F Orive R (2007); Székelyhidi L, Jr (2012); Castro A, Córdoba D, Fefferman C, Gancedo F, López-Fernández M (2012); Muskat M (1934); Saffman PG, Taylor G (1958); Siegel M, Caflisch R, Howison S (2004); Escher J, Matioc BV (2011); Córdoba D, Gancedo F (2007); Ambrose DM (2004); Córdoba A, Córdoba D, Gancedo F (2011); Lannes D (2013); Constantin P, Córdoba D, Gancedo F, Strain RM (2013); Beck T, Sosoe P, Wong P (2014); Castro A, Córdoba D, Fefferman C, Gancedo F (2013); Wu S (1997); Wu S (2009); Ionescu AD, Pusateri F (2013); Alazard T, Delort JM (2013); Castro A, Córdoba D, Fefferman C, Gancedo F, Gómez-Serrano J (2012); Castro A, Córdoba D, Fefferman D, Gancedo F, Gómez-Serrano J. (2014); C. Fefferman, A. Ionescu and V. Lie (2014); Coutand D, Shkoller S (2013); Córdoba D, Gancedo F (2010); Escher J, Matioc AV, Matioc BV (2012); Constantin A, Escher J (1998); Córdoba A, Córdoba D (2003); Constantin, Gancedo, Shvydkoy, Vicol (2017) ...

The linearized equation when $A_\mu = 0$ in $\mathbb{R}$

This equation for f can be linearized around the flat solution:

$$f_t^L(\alpha, t) = -\frac{\rho^2 - \rho^1}{2}\Lambda(f^L)(\alpha, t), \qquad \Lambda = (-\Delta)^{1/2}.$$

The linearized equation when $A_\mu = 0$ in $\mathbb{R}$

This equation for f can be linearized around the flat solution:

$$f^L_t(\alpha,t) = -\frac{\rho^2-\rho^1}{2}\Lambda(f^L)(\alpha,t), \qquad \Lambda = (-\Delta)^{1/2}.$$

The linearized equation can be solved by Fourier transform:

$$\hat{f}^L(\xi) = \hat{f}_0(\xi)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi|t\Big).$$

The linearized equation when $A_\mu = 0$ in $\mathbb{R}$

This equation for f can be linearized around the flat solution:

$$f_t^L(\alpha,t) = -\frac{\rho^2-\rho^1}{2}\Lambda(f^L)(\alpha,t), \qquad \Lambda = (-\Delta)^{1/2}.$$

The linearized equation can be solved by Fourier transform:

$$\hat{f}^L(\xi) = \hat{f}_0(\xi)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi|t\Big).$$

- $\rho^2 > \rho^1$ stable case, we have well-posedness.
- $\rho^2 < \rho^1$ unstable case, we have ill-posedness.
  See Ambrose (2004), Córdoba & Gancedo (2007), ...

The linearized equation when $A_\mu = 0$ in $\mathbb{R}$

This equation for f can be linearized around the flat solution:

$$f_t^L(\alpha,t) = -\frac{\rho^2-\rho^1}{2}\Lambda(f^L)(\alpha,t), \qquad \Lambda = (-\Delta)^{1/2}.$$

The linearized equation can be solved by Fourier transform:

$$\hat{f}^L(\xi) = \hat{f}_0(\xi)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi| t\Big).$$

- $\rho^2 > \rho^1$ stable case, we have well-posedness.
- $\rho^2 < \rho^1$ unstable case, we have ill-posedness.
  See Ambrose (2004), Córdoba & Gancedo (2007), ...
- Also we have the L^2 evolution for the linear equation:

$$\frac{d}{dt}\|f^L\|_{L^2}^2(t) = -\frac{\rho^2-\rho^1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\frac{f^L(\alpha,t)-f^L(\beta,t)}{\alpha-\beta}\Big)^2 d\alpha d\beta dt.$$

This is a smoothing estimate. Similar in 3D.

Smoothing for the non-linear equation?

$$f_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} d\beta \frac{(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\alpha-\beta,t))\,\beta}{\beta^2 + (f(\alpha,t) - f(\alpha-\beta,t))^2}.$$

Smoothing for the non-linear equation?

$$f_t(\alpha,t) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} d\beta \frac{(\partial_\alpha f(\alpha,t) - \partial_\alpha f(\alpha-\beta,t))\,\beta}{\beta^2 + (f(\alpha,t)-f(\alpha-\beta,t))^2}.$$

Satisfies L^2 maximum principle:

$$\frac{d}{dt}\|f\|^2_{L^2}(t) = -\frac{\rho^2-\rho^1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\left(1+\left(\frac{f(\alpha,t)-f(\beta,t)}{\alpha-\beta}\right)^2\right) d\alpha d\beta$$

Smoothing for the non-linear equation?

$$f_t(\alpha,t)=\frac{\rho^2-\rho^1}{2\pi}PV\int_{\mathbb{R}}d\beta\frac{(\partial_\alpha f(\alpha,t)-\partial_\alpha f(\alpha-\beta,t))\,\beta}{\beta^2+(f(\alpha,t)-f(\alpha-\beta,t))^2}.$$

Satisfies L^2 maximum principle:

$$\frac{d}{dt}\|f\|^2_{L^2}(t)=-\frac{\rho^2-\rho^1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\ln\left(1+\left(\frac{f(\alpha,t)-f(\beta,t)}{\alpha-\beta}\right)^2\right)d\alpha d\beta$$

For which it is possible to bound as follows:

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\ln\left(1+\left(\frac{f(\alpha,t)-f(\beta,t)}{\alpha-\beta}\right)^2\right)d\alpha d\beta\leq 4\pi\sqrt{2}\|f\|_{L^1}(t).$$

Smoothing for the non-linear equation?

$$f_t(\alpha, t) = \frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} d\beta \frac{(\partial_\alpha f(\alpha, t) - \partial_\alpha f(\alpha - \beta, t))\, \beta}{\beta^2 + (f(\alpha, t) - f(\alpha - \beta, t))^2}.$$

Satisfies L^2 maximum principle:

$$\frac{d}{dt}\|f\|_{L^2}^2(t) = -\frac{\rho^2 - \rho^1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \ln\left(1 + \left(\frac{f(\alpha, t) - f(\beta, t)}{\alpha - \beta}\right)^2\right) d\alpha d\beta$$

For which it is possible to bound as follows:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln\left(1 + \left(\frac{f(\alpha, t) - f(\beta, t)}{\alpha - \beta}\right)^2\right) d\alpha d\beta \le 4\pi\sqrt{2}\|f\|_{L^1}(t).$$

Don’t see a non-linear smoothing effect at the level of f in L^2. See P. Constantin, D. Córdoba, F. Gancedo - S. (2013). Also a similar “no-smoothing” statement in 3D. And Córdoba, J. Gómez-Serrano and A. Zlatos (2017)

- Maximum Principle: Constantin, Córdoba-Gancedo, S (2013) in 2D:

$$\|\nabla f_0\|_{L^\infty(\mathbb{R})} < 1 \implies \|\nabla f\|_{L^\infty(\mathbb{R})}(t) < 1.$$

  Then Constantin, Córdoba, Gancedo, Rodriguez-Piazza, S (2016) in 3D:

$$\|\nabla f_0\|_{L^\infty(\mathbb{R}^2)} < 1/3 \implies \|\nabla f\|_{L^\infty(\mathbb{R}^2)}(t) < 1/3.$$

  These statements allow you to conclude the global existence of weak solutions.

Theorem (Constantin-Córdoba-Gancedo- Piazza- S (2016))

In 2D ($d = 1$) we suppose for some $0 < \delta < 1$ that

$$\int |\xi|^1 |\widehat{f_0}(\xi)| d\xi, \leq c_0, \quad 2\sum_{n\geq 1}(2n+1)^{1+\delta} c_0^{2n} \leq 1, \quad c_0 \geq \frac{1}{3}$$

Then there is a unique Muskat solution with initial data f_0 that satisfies $f \in C([0,T]; H^l(\mathbb{R}^d))$ for any $T > 0$.

A few recent papers

- Constantin, Gancedo, Shvydkoy, Vicol (2017): Local well posedness for initial data with finite slope. Further the solution has a continuation criterion as long as $\|f_0'\|_{L^\infty} < \infty$. Also global well posedness for initial data with very small slope:

$$f_0 \in L^2(\mathbb{R}),\ f_0'' \in L^p(\mathbb{R}), 1 < p \leq \infty, \quad \|f_0'\|_{L^\infty} \ll 1$$

A few recent papers

- Constantin, Gancedo, Shvydkoy, Vicol (2017): Local well posedness for initial data with finite slope. Further the solution has a continuation criterion as long as $\|f_0'\|_{L^\infty} < \infty$. Also global well posedness for initial data with very small slope:

$$f_0 \in L^2(\mathbb{R}),\ f_0'' \in L^p(\mathbb{R}), 1 < p \leq \infty, \quad \|f_0'\|_{L^\infty} \ll 1$$

- Matioc (2017): Well posedness 2D ($d = 1$) for initial data $f_0 \in H^l(\mathbb{R})$ for $l \in (3/2, 2)$. (with surface tension for $l \in (2, 3)$.)

A few recent papers

- Constantin, Gancedo, Shvydkoy, Vicol (2017): Local well posedness for initial data with finite slope. Further the solution has a continuation criterion as long as $\|f_0'\|_{L^\infty} < \infty$. Also global well posedness for initial data with very small slope:

$$f_0 \in L^2(\mathbb{R}),\ f_0'' \in L^p(\mathbb{R}), 1 < p \leq \infty, \quad \|f_0'\|_{L^\infty} \ll 1$$

- Matioc (2017): Well posedness 2D ($d = 1$) for initial data $f_0 \in H^l(\mathbb{R})$ for $l \in (3/2, 2)$. (with surface tension for $l \in (2, 3)$.)
- Deng, Lei, Lin (2017) In 2D, prove the existence of global weak solutions for arbitrarily large monotonic initial data.

A few recent papers

- Constantin, Gancedo, Shvydkoy, Vicol (2017): Local well posedness for initial data with finite slope. Further the solution has a continuation criterion as long as $\|f_0'\|_{L^\infty} < \infty$. Also global well posedness for initial data with very small slope:

$$f_0 \in L^2(\mathbb{R}),\ f_0'' \in L^p(\mathbb{R}), 1 < p \leq \infty, \quad \|f_0'\|_{L^\infty} \ll 1$$

- Matioc (2017): Well posedness 2D ($d = 1$) for initial data $f_0 \in H^l(\mathbb{R})$ for $l \in (3/2, 2)$. (with surface tension for $l \in (2, 3)$.)
- Deng, Lei, Lin (2017) In 2D, prove the existence of global weak solutions for arbitrarily large monotonic initial data.
- Cameron (2017), in 2D, introduces a modulus of continuity and proves that weak solutions are in fact $C^0_{t,\alpha} \cap C^{1,\gamma}_{loc,t,\alpha}$. And they are unique if $f_0 \in C^{1,\epsilon}$.

Muskat problem with Viscosity Jump $A_\mu \neq 0$

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

$$\mu(x,t)u(x,t) = -\nabla p(x,t) - \rho(x,t)e_3,$$

The densities and viscosities of each fluid are given by

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t). \end{cases}$$

The open sets $D^1(t)$ and $D^2(t)$ are connected and move with the velocity of the fluid

$$\frac{dx}{dt}(t) = u(x(t),t), \quad \forall\, x(t) \in D^j(t), \text{ or } x(t) \in \partial D^j(t).$$

Contour Equation with Viscosity Jump $A_\mu \neq 0$

Then the evolution equation for the interface in the Eulerian-Lagrangian view is

$$\partial D^j(t) = \{X(\alpha, t) : \alpha \in \mathbb{R}^2\}$$

where

$$\partial_t X(\alpha, t) = BR(X, \omega)(\alpha, t) + C_1(\alpha, t)\partial_{\alpha_1} X(\alpha, t) + C_2(\alpha, t)\partial_{\alpha_2} X(\alpha, t),$$

where *BR* is the well-known Birkhoff-Rott integral

$$BR(X, \omega)(\alpha, t) = -\frac{1}{4\pi} PV \int_{\mathbb{R}^2} \frac{X(\alpha, t) - X(\beta, t)}{|X(\alpha, t) - X(\beta, t)|^3} \wedge \omega(\beta, t) d\beta.$$

The “constants” C_1 and C_2 can be chosen. And ω is the amplitude of the vorticity.

Vorticity Equations

The vorticity ω is related to the potential jump $\Omega(\alpha, t)$ by

$$\omega(\alpha, t) = \partial_{\alpha_2}\Omega(\alpha, t)\partial_{\alpha_1}X(\alpha, t) - \partial_{\alpha_1}\Omega(\alpha, t)\partial_{\alpha_2}X(\alpha, t).$$

The potential jump is given implicitly by

$$\Omega(\alpha, t) = A_\mu \mathcal{D}(\Omega)(\alpha, t) - 2A_\rho X_3(\alpha, t),\ A_\mu = \frac{\mu^2 - \mu^1}{\mu^2 + \mu^1},\ A_\rho = \frac{\rho^2 - \rho^1}{\rho^2 + \rho^1},$$

where $\mathcal{D}$ is the double layer potential

$$\mathcal{D}(\Omega)(\alpha, t) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \frac{X(\alpha, t) - X(\beta, t)}{|X(\alpha, t) - X(\beta, t)|^3} \cdot N(\beta, t)\Omega(\beta, t)d\beta.$$

Here $N(\alpha, t) = \partial_{\alpha_1}X(\alpha, t) \wedge \partial_{\alpha_2}X(\alpha, t)$.

Recall the Atwood Number is A_μ: And we observe that $A_\mu = 0$ simplifies the situation dramatically, the equation is more local.

Evolution Equation for the interface

When the evolving interface can be described as a graph

$$X(\alpha,t) = (\alpha_1, \alpha_2, f(\alpha,t)), \qquad \alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2$$

then the equations can be reduced to one as follows

$$\begin{aligned} f_t(\alpha) = &-\frac{1}{4\pi}\int_{\mathbb{R}^2} \frac{(\alpha_1-\beta_1)\omega_2(\beta)-(\alpha_2-\beta_2)\omega_1(\beta)}{|(\alpha_1,\alpha_2,f(\alpha))-(\beta_1,\beta_2,f(\beta))|^3}d\beta \\ &+ C_1(\alpha)\partial_{\alpha_1}f(\alpha) + C_2(\alpha)\partial_{\alpha_2}f(\alpha). \end{aligned}$$

Evolution Equation for the interface

When the evolving interface can be described as a graph

$$X(\alpha,t)=(\alpha_1,\alpha_2,f(\alpha,t)),\qquad \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$$

then the equations can be reduced to one as follows

$$f_t(\alpha)=-\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{(\alpha_1-\beta_1)\omega_2(\beta)-(\alpha_2-\beta_2)\omega_1(\beta)}{|(\alpha_1,\alpha_2,f(\alpha))-(\beta_1,\beta_2,f(\beta))|^3}d\beta$$
$$+C_1(\alpha)\partial_{\alpha_1}f(\alpha)+C_2(\alpha)\partial_{\alpha_2}f(\alpha).$$

We further obtain the coefficients as

$$C_1(\alpha)=\frac{1}{4\pi}\mathrm{PV}\int_{\mathbb{R}^2}\frac{(\alpha_2-\beta_2)\omega_3(\beta)-\omega_2(\beta)(f(\alpha)-f(\beta))}{|(\alpha_1,\alpha_2,f(\alpha))-(\beta_1,\beta_2,f(\beta))|^3}d\beta,$$
$$C_2(\alpha)=\frac{1}{4\pi}\mathrm{PV}\int_{\mathbb{R}^2}\frac{\omega_1(\beta)(f(\alpha)-f(\beta))-(\alpha_1-\beta_1)\omega_3(\beta)}{|(\alpha_1,\alpha_2,f(\alpha))-(\beta_1,\beta_2,f(\beta))|^3}d\beta.$$

Revealing the Parabolic structure when $A_\mu \neq 0$

After several further calculations we can find the equations

$$f_t = -A_\rho \Lambda f + N(f), \qquad \text{where} \qquad N(f) = N_1(f) + N_2(f) + N_3(f),$$

Above we observe the smoothing operator Λf. Here $N(f) = N(f, \Omega)$ and

$$N_1 = \frac{A_\mu}{2}\Lambda \mathcal{D}(\Omega)(\alpha),$$

$$N_2 = \frac{1}{4\pi}\mathrm{PV}\int \left(\frac{\frac{\beta}{|\beta|} + \Delta_\beta f(\alpha)\nabla f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}} - \frac{\beta}{|\beta|} \right) \cdot \frac{\nabla\Omega(\alpha-\beta)}{|\beta|^2} d\beta,$$

$$N_3 = \frac{A_\mu}{4\pi}\mathrm{PV}\int_{\mathbb{R}^2} \frac{\beta \cdot \nabla^\perp f(\alpha)\nabla\mathcal{D}(\Omega)(\alpha-\beta) \cdot \nabla^\perp f(\alpha-\beta)}{(1+(\Delta_\beta f(\alpha))^2)^{\frac{3}{2}}} \frac{d\beta}{|\beta|^3}.$$

Now we can see some smoothing in the equations at the level of f plus complicated non-linear terms.

A few references when $A_\mu \neq 0$

- (Córdoba, Córdoba, Gancedo (2013)) For a general curve, Local well-posedness in Sobolev spaces if the initial interface satisfies the Rayleigh-Taylor condition:

$$(\nabla p^2 - \nabla p^1) \cdot (\nu^2 - \nu^1) > 0$$

where ν^j is inner unit normal to domain D^j.

A few references when $A_\mu \neq 0$

- (Córdoba, Córdoba, Gancedo (2013)) For a general curve, Local well-posedness in Sobolev spaces if the initial interface satisfies the Rayleigh-Taylor condition:

$$(\nabla p^2 - \nabla p^1) \cdot (\nu^2 - \nu^1) > 0$$

  where ν^j is inner unit normal to domain D^j.
- Cheng, Granero-Belinchón and Shkoller (2016): global in time classical solutions for small initial data in subcritical initially small H^2 norms in 2D. $A_\mu \neq 0$.

Gain of Regularity also when $A_\mu \neq 0$

Define the analytic norms for $\nu > 0$:

$$\|f\|_{\mathcal{F}^{s,p}_\nu} = \|e^{t\nu|\xi|}|\xi|^s \hat{f}(\xi)\|_{L^p_\xi}$$

where in the case $p = 2$, we call this space H^s_ν.

Gain of Regularity also when $A_\mu \neq 0$

Define the analytic norms for $\nu > 0$:

$$\|f\|_{\mathcal{F}^{s,p}_\nu} = \|e^{t\nu|\xi|}|\xi|^s \hat{f}(\xi)\|_{L^p_\xi}$$

where in the case $p = 2$, we call this space H^s_ν.

Theorem (Gancedo, García-Juárez, Patel, S (2017): GGPS)

We can prove global existence and uniqueness in $\mathcal{F}^{1,1} \cap L^2$ with medium size initial data in the stable case allowing $\mu_1 \neq \mu_2$. Moreover, the solution is instantly analytic; there exists a $\nu > 0$ such that for $s \geq 0$ there exist constants $C_s, \tilde{C}_s > 0$ such that

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{s,1}_\nu}(t) \leq -C_s\|f\|_{\mathcal{F}^{s+1,1}_\nu}(t) \text{ and } \|f\|_{\mathcal{F}^{s,1}_\nu}(t) \leq \tilde{C}_s$$

for $t \geq T_s$. For $0 \leq s \leq 1$, we can take $T_s = 0$.

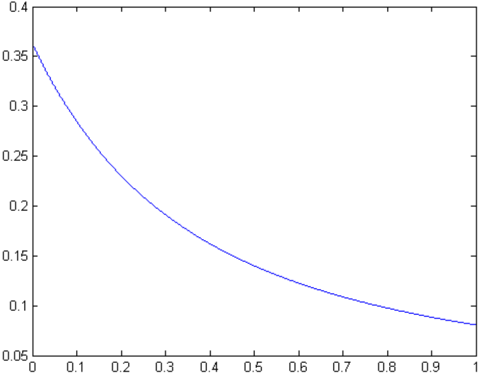
We have a similar statement for $\|f\|_{H^r_\nu}(t)$.

Size of $k(|A_\mu|) > 0$

We assume initially that

$$\|f_0\|_{\mathcal{F}_0^{1,1}} = \||\xi|\hat{f}_0(\xi)\|_{L^1_\xi} < k(|A_\mu|), \quad \|f_0\|_{L^2} < \infty.$$

Below we give a numerical estimate of the size of the constant $k(|A_\mu|)$ for $0 \leq |A_\mu| \leq 1$. We need $k(|A_\mu|)$ to be small enough to make a high order rational polynomial be positive.



Instant Analyticity

By introducing the formulas for the vorticity ω into I_1, I_2 and I_3, we can express the interface equation as

$$f_t(\alpha) = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3$$

where

$$\tilde{I}_1 = -\frac{A_\rho}{2\pi}\Lambda f(\alpha) - \frac{A_\mu}{4\pi}\Lambda D(\Omega)(\alpha),$$

in which

$$D(\Omega)(\alpha) = \frac{1}{2\pi}\int \frac{\frac{\beta_1}{|\beta|}\partial_{\alpha_1} f(\alpha-\beta) + \frac{\beta_2}{|\beta|}\partial_{\alpha_2} f(\alpha-\beta) - \Delta_\beta f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}} \frac{\Omega(\alpha-\beta)}{|\beta|^2},$$

$$\tilde{I}_2 = \sum_{i=1}^{2} \frac{1}{4\pi}\int \left(\frac{\frac{\beta_i}{|\beta|} + \Delta_\beta f(\alpha)\partial_{\alpha_i} f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}} - \frac{\beta_i}{|\beta|} \right) \frac{\partial_{\alpha_i}\Omega(\alpha-\beta)}{|\beta|^2} d\beta$$

and

$$\tilde{I}_3 = \frac{1}{4\pi}\int \frac{\frac{\beta_2}{|\beta|}\partial_{\alpha_1} f(\alpha) - \frac{\beta_1}{|\beta|}\partial_{\alpha_2} f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}} \frac{\omega_3(\alpha-\beta)}{|\beta|^2} d\beta.$$

Differentiating in time:

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{s,1}_{\nu}}(t) = \nu \int |\xi|^{s+1} e^{t\nu|\xi|}|\hat{f}(\xi)|d\xi + \int |\xi|^{s} e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\hat{f}_t\overline{\hat{f}} + \hat{f}\overline{\hat{f}_t}}{|\hat{f}(\xi)|}\Big)d\xi$$

Differentiating in time:

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{s,1}_\nu}(t) = \nu\int |\xi|^{s+1}e^{t\nu|\xi|}|\hat{f}(\xi)|d\xi + \int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\hat{f}_t\overline{\hat{f}} + \hat{f}\overline{\hat{f}_t}}{|\hat{f}(\xi)|}\Big)d\xi$$

The linear term in the equation gives

$$-\frac{A_\rho}{2\pi}\int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\frac{|\xi|\hat{f}\overline{\hat{f}} + \hat{f}|\xi|\overline{\hat{f}}}{|\hat{f}(\xi)|}d\xi = -\frac{A_\rho}{2\pi}\int |\xi|^{s+1}e^{t\nu|\xi|}|\hat{f}(\xi)|d\xi$$
$$= -\frac{A_\rho}{2\pi}\|f\|_{\mathcal{F}^{s+1,1}_\nu}$$

Differentiating in time:

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{s,1}_\nu}(t) = \nu \int |\xi|^{s+1} e^{t\nu|\xi|}|\hat{f}(\xi)|d\xi + \int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\hat{f}_t\bar{\hat{f}} + \hat{f}\bar{\hat{f}}_t}{|\hat{f}(\xi)|}\Big)d\xi$$

The linear term in the equation gives

$$\begin{aligned} -\frac{A_\rho}{2\pi}\int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\frac{|\xi|\hat{f}\bar{\hat{f}} + \hat{f}|\xi|\bar{\hat{f}}}{|\hat{f}(\xi)|}d\xi &= -\frac{A_\rho}{2\pi}\int |\xi|^{s+1} e^{t\nu|\xi|}|\hat{f}(\xi)|d\xi \\ &= -\frac{A_\rho}{2\pi}\|f\|_{\mathcal{F}^{s+1,1}_\nu} \end{aligned}$$

then

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{s,1}_\nu}(t) \leq \left(\nu - \frac{A_\rho}{2\pi}\right)\|f\|_{\mathcal{F}^{s+1,1}_\nu} + \text{non-linear terms}$$

It remains to bound the nonlinear terms of the evolution by the negative linear growth: $\left(\nu - \frac{A_\rho}{2\pi}\right) < 0$.

We consider the one of the terms in $\tilde{I}_2$ as an example.

We consider the one of the terms in $\tilde{I}_2$ as an example. Using Taylor expansion, since $\|\Delta_\beta f\|_{L^\infty} \leq \|\nabla f\|_{L^\infty} \leq \|f\|_{\mathcal{F}^{1,1}} < 1$:

$$
\begin{aligned}
I_{2,1} &= \int \frac{\Delta_\beta f(\alpha)\partial_{\alpha_i} f(\alpha)}{(1+(\Delta_\beta f(\alpha))^2)^{3/2}} \frac{\partial_{\alpha_i}\Omega(\alpha-\beta)}{|\beta|^2} d\beta \\
&= \int \sum_{n\geq 0} a_n \left(\Delta_\beta f(\alpha)\right)^{2n+1} \partial_{\alpha_i}\Omega(\alpha-\beta)\partial_{\alpha_i} f(\alpha) \frac{d\beta}{|\beta|^2}
\end{aligned}
$$

We consider the one of the terms in $\tilde{I}_2$ as an example. Using Taylor expansion, since $\|\Delta_\beta f\|_{L^\infty} \leq \|\nabla f\|_{L^\infty} \leq \|f\|_{\mathcal{F}^{1,1}} < 1$:

$$
\begin{aligned}
I_{2,1} &= \int \frac{\Delta_\beta f(\alpha) \partial_{\alpha_i} f(\alpha)}{(1 + (\Delta_\beta f(\alpha))^2)^{3/2}} \frac{\partial_{\alpha_i} \Omega(\alpha - \beta)}{|\beta|^2} d\beta \\
&= \int \sum_{n \geq 0} a_n \left(\Delta_\beta f(\alpha)\right)^{2n+1} \partial_{\alpha_i} \Omega(\alpha - \beta) \partial_{\alpha_i} f(\alpha) \frac{d\beta}{|\beta|^2}
\end{aligned}
$$

Taking the Fourier transform, we can show the estimate

$$
|\widehat{I_{2,1}}| \leq \sum_{n \geq 0} a_n |\widehat{\partial_{\alpha_i} \Omega}(\cdot)| *^{2n+2} (|\cdot||\hat{f}|(\cdot))
$$

a_n are the coefficients from Taylor expansion of $1/(1 + x^2)^{3/2}$ and $*^{2n+2}(g)$ represents $2n + 2$ iterated convolutions of g.

Plugging this into the evolution inequality for $\|f\|_{\mathcal{F}^{s,1}_\nu}$:

$$
\begin{aligned}
\int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\hat{f}_t\bar{\hat{f}}+\hat{f}\bar{\hat{f}}_t}{|\hat{f}(\xi)|}\Big)d\xi &\le \int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\widehat{I_{2,1}}\bar{\hat{f}}+\hat{f}\overline{\widehat{I_{2,1}}}}{|\hat{f}(\xi)|}\Big)d\xi+\cdots\\
&\le \int |\xi|^s e^{t\nu|\xi|}|\widehat{I_{2,1}}(\xi)|d\xi+\cdots\\
&\le \sum_{n\ge 0} a_n \int |\xi|^s\Big(e^{t\nu|\xi|}|\widehat{\partial_{\alpha_1}\Omega}(\xi)| *^{2n+2} (|\xi||\hat{f}|(\xi))\Big)+\cdots
\end{aligned}
$$

Plugging this into the evolution inequality for $\|f\|_{\mathcal{F}^{s,1}_\nu}$:

$$\int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\hat{f}_t\bar{\hat{f}}+\hat{f}\bar{\hat{f}_t}}{|\hat{f}(\xi)|}\Big)d\xi \le \int |\xi|^s e^{t\nu|\xi|}\frac{1}{2}\Big(\frac{\widehat{I_{2,1}}\bar{\hat{f}}+\hat{f}\overline{\widehat{I_{2,1}}}}{|\hat{f}(\xi)|}\Big)d\xi+\cdots$$
$$\le \int |\xi|^s e^{t\nu|\xi|}|\widehat{I_{2,1}}(\xi)|d\xi+\cdots$$
$$\le \sum_{n\ge 0} a_n\int |\xi|^s\Big(e^{t\nu|\xi|}|\widehat{\partial_{\alpha_1}\Omega}(\xi)|*^{2n+2}(|\xi||\hat{f}|(\xi))\Big)+\cdots$$

We can now apply the triangle inequality to distribute the exponential and the multiplier $|\xi|^s$ to each term to obtain

$$\le \sum_{n\ge 0} a_n\int (|\xi|^s e^{t\nu|\xi|}|\widehat{\partial_{\alpha_1}\Omega}(\xi)|)*^{2n+2}(|\xi|e^{t\nu|\xi|}|\hat{f}|(\xi))$$
$$+(2n+2)(|e^{t\nu|\xi|}\widehat{\partial_{\alpha_1}\Omega}(\xi)|)*^{2n+1}(|\xi|e^{t\nu|\xi|}|\hat{f}|(\xi))*(|\xi|^{s+1}e^{t\nu|\xi|}|\hat{f}|(\xi))d\xi$$

From here, we apply Young's inequality to obtain that this term is bounded by

$$\leq \sum_{n\geq 0} a_n \|\partial_{\alpha_1}\Omega\|_{\mathcal{F}_\nu^{s,1}} \|f\|_{\mathcal{F}_\nu^{1,1}}^{2n+2} + (2n+2)\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}_\nu^{0,1}} \|f\|_{\mathcal{F}_\nu^{1,1}}^{2n+1} \|f\|_{\mathcal{F}_\nu^{s+1,1}}$$

From here, we apply Young's inequality to obtain that this term is bounded by

$$\leq \sum_{n\geq 0} a_n \|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{s,1}_{\nu}} \|f\|^{2n+2}_{\mathcal{F}^{1,1}_{\nu}} + (2n+2)\|\partial_{\alpha_1}\Omega\|_{\mathcal{F}^{0,1}_{\nu}} \|f\|^{2n+1}_{\mathcal{F}^{1,1}_{\nu}} \|f\|_{\mathcal{F}^{s+1,1}_{\nu}}$$

Using the equations for the vorticity $\partial_{\alpha_1}\Omega$ can be bounded

$$\|\partial_{\alpha_i}\Omega\|_{\mathcal{F}^{0,1}_{\nu}} \leq B_\mu \text{ and } \|\partial_{\alpha_i}\Omega\|_{\mathcal{F}^{s,1}_{\nu}} \leq \tilde{B}_\mu \|f\|_{\mathcal{F}^{s+1,1}_{\nu}}$$

The constants $B_\mu, \tilde{B}_\mu \to 0$ as $\|f\|_{\mathcal{F}^{1,1}_{\nu}} \to 0$ and are uniform in ν.

Hence, for $\|f\|_{\mathcal{F}_\nu^{1,1}}(t)$ of medium size depending on

$$A_\mu = (\mu_2 - \mu_1)/(\mu_2 + \mu_1)$$

and $\nu > 0$ small enough, the nonlinear terms are sufficiently small to conclude

$$\frac{d}{dt}\|f\|_{\mathcal{F}_\nu^{s,1}}(t) \leq -C_\mu \|f\|_{\mathcal{F}_\nu^{s+1,1}}(t)$$

for a positive constant C_μ and $0 \leq s \leq 1$.

Gain of L^2 Analyticity

It suffices to perform estimates on $\|f\|_{L^2_\nu}$ to instantly gain regularity in H^s:

$$\|f\|_{H^s} \leq \|(1+|\xi|^2)^{s/2} e^{-t\nu|\xi|}\|_{L^\infty} \|f\|_{L^2_\nu}$$

Gain of L^2 Analyticity

It suffices to perform estimates on $\|f\|_{L^2_\nu}$ to instantly gain regularity in H^s:

$$\|f\|_{H^s} \leq \|(1+|\xi|^2)^{s/2} e^{-t\nu|\xi|}\|_{L^\infty} \|f\|_{L^2_\nu}$$

Theorem

Suppose $f_0 \in L^2 \cap \mathcal{F}^{1,1}$ and $\|f_0\|_{\mathcal{F}^{1,1}}$ satisfying the medium size condition. Then, $f(t) \in L^2_\nu$ instantly for all $t > 0$. Moreover, this implies that $f(t) \in H^s$ for any $s > 0$ instantly for all $t > 0$.

Gain of L^2 Analyticity

It suffices to perform estimates on $\|f\|_{L^2_\nu}$ to instantly gain regularity in H^s:

$$\|f\|_{H^s} \leq \|(1+|\xi|^2)^{s/2} e^{-t\nu|\xi|}\|_{L^\infty} \|f\|_{L^2_\nu}$$

Theorem

Suppose $f_0 \in L^2 \cap \mathcal{F}^{1,1}$ and $\|f_0\|_{\mathcal{F}^{1,1}}$ satisfying the medium size condition. Then, $f(t) \in L^2_\nu$ instantly for all $t > 0$. Moreover, this implies that $f(t) \in H^s$ for any $s > 0$ instantly for all $t > 0$.

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_\nu}(t) = (\nu - \frac{A_\rho}{4\pi})\|f\|^2_{\dot{H}^{1/2}_\nu} + \int \frac{A_\mu}{4\pi}|\xi| e^{2\nu t|\xi|}|\widehat{D(\Omega)}(\xi)||\hat{f}(\xi)|d\xi$$

$$+ \int e^{2\nu t|\xi|}|\widehat{\tilde{I}_2}(\xi)||\hat{f}(\xi)|d\xi + \int e^{2\nu t|\xi|}|\widehat{\tilde{I}_3}(\xi)||\hat{f}(\xi)|d\xi$$

We now bound the nonlinear terms. For example,

$$\int |\xi| e^{2\nu t|\xi|} |\widehat{D(\Omega)}(\xi)||\hat{f}(\xi)| d\xi \leq \|f\|_{\dot{H}^{1/2}_{\nu}} \|D(\Omega)\|_{\dot{H}^{1/2}_{\nu}}$$

We now bound the nonlinear terms. For example,

$$\int |\xi| e^{2\nu t|\xi|} |\widehat{D(\Omega)}(\xi)||\hat{f}(\xi)| d\xi \leq \|f\|_{\dot{H}^{1/2}_\nu} \|D(\Omega)\|_{\dot{H}^{1/2}_\nu}$$

and using iterated Young's inequality we have

$$\begin{aligned}
&\int e^{2\nu t|\xi|} |\widehat{\tilde{I}^{1,1}_2}(\xi)||\hat{f}(\xi)| d\xi \\
&\quad \leq \frac{1}{2} \sum_{n\geq 0} a_{n+1} \int e^{2\nu t|\xi|} |\hat{f}(\xi)||\widehat{\partial_{\alpha_1}\Omega}(\cdot)| * (*^{2n+2}|\cdot||\hat{f}(\cdot)|) d\xi \\
&\leq \frac{1}{2} \sum_{n\geq 0} a_{n+1} \int e^{\nu t|\xi|} |\xi||\widehat{\Omega}(\xi)| \cdot (e^{\nu t|\cdot|}|\hat{f}(\cdot)|) * (*^{2n+2}|\cdot| e^{\nu t|\cdot|}|\hat{f}(\cdot)|) d\xi \\
&\leq \sum_{n\geq 0} (n+1) a_{n+1} \frac{\epsilon}{2} \|\Omega\|^2_{\dot{H}^{1/2}_\nu} + (n+1) a_{n+1} \frac{1}{2\epsilon} \|f\|^2_{L^2_\nu} \|f\|^2_{\mathcal{F}^{3/2,1}_\nu} \|f\|^{4n+2}_{\mathcal{F}^{1,1}_\nu} \\
&\qquad + \frac{1}{2} a_{n+1} \|\Omega\|_{\dot{H}^{1/2}_\nu} \|f\|_{\dot{H}^{1/2}_\nu} \|f\|^{2n+2}_{\mathcal{F}^{1,1}_\nu}
\end{aligned}$$

It remains to bound $\|D(\Omega)\|_{\dot{H}^{1/2}_{\nu}}$ and $\|\Omega\|_{\dot{H}^{1/2}_{\nu}}$ appropriately:

It remains to bound $\|D(\Omega)\|_{\dot{H}^{1/2}_\nu}$ and $\|\Omega\|_{\dot{H}^{1/2}_\nu}$ appropriately:

$$\|D(\Omega)\|_{\dot{H}^{1/2}_\nu} \leq \sum_{n\geq 0} a_n \| |\xi|^{\frac{1}{2}} e^{\nu t|\xi|} (*^{2n+1} |\cdot||\hat{f}(\cdot)|) * |\widehat{\Omega}(\cdot)| \|_{L^2_\nu}$$

$$\leq \sum_{n\geq 0} a_n \|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{\dot{H}^{1/2}_\nu} + (2n+1) a_n \|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|^{2n}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{L^2_\nu},$$

It remains to bound $\|D(\Omega)\|_{\dot{H}^{1/2}_\nu}$ and $\|\Omega\|_{\dot{H}^{1/2}_\nu}$ appropriately:

$$\|D(\Omega)\|_{\dot{H}^{1/2}_\nu} \le \sum_{n\ge 0} a_n \||\xi|^{\frac{1}{2}} e^{\nu t|\xi|}(*^{2n+1}|\cdot||\hat{f}(\cdot)|) * |\widehat{\Omega}(\cdot)|\|_{L^2_\nu}$$

$$\le \sum_{n\ge 0} a_n \|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{\dot{H}^{1/2}_\nu} + (2n+1) a_n \|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|^{2n}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{L^2_\nu},$$

which implies

$$\|\Omega\|_{\dot{H}^{1/2}_\nu} \le (1 - A_\mu \sum_{n\ge 0} a_n \|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu})^{-1}$$

$$\cdot \Big(A_\mu \sum_{n\ge 0} (2n+1) a_n \|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|^{2n}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{L^2_\nu} + 2A_\rho \|f\|_{\dot{H}^{1/2}_\nu} \Big)$$

It remains to bound $\|D(\Omega)\|_{\dot{H}^{1/2}_\nu}$ and $\|\Omega\|_{\dot{H}^{1/2}_\nu}$ appropriately:

$$\|D(\Omega)\|_{\dot{H}^{1/2}_\nu} \leq \sum_{n\geq 0} a_n \| |\xi|^{\frac{1}{2}} e^{\nu t|\xi|} (*^{2n+1} |\cdot||\hat{f}(\cdot)|) * |\widehat{\Omega}(\cdot)| \|_{L^2_\nu}$$
$$\leq \sum_{n\geq 0} a_n \|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{\dot{H}^{1/2}_\nu} + (2n+1) a_n \|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|^{2n}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{L^2_\nu},$$

which implies

$$\|\Omega\|_{\dot{H}^{1/2}_\nu} \leq (1 - A_\mu \sum_{n\geq 0} a_n \|f\|^{2n+1}_{\mathcal{F}^{1,1}_\nu})^{-1}$$
$$\cdot \Big(A_\mu \sum_{n\geq 0} (2n+1) a_n \|f\|_{\mathcal{F}^{3/2,1}_\nu} \|f\|^{2n}_{\mathcal{F}^{1,1}_\nu} \|\Omega\|_{L^2_\nu} + 2A_\rho \|f\|_{\dot{H}^{1/2}_\nu} \Big)$$

Hence

$$\|D(\Omega)\|_{\dot{H}^{1/2}_\nu} \leq C(\|f\|_{\mathcal{F}^{1,1}}) \Big(\|f\|_{\mathcal{F}^{3/2,1}_\nu} \|\Omega\|_{L^2_\nu} + \|f\|_{\dot{H}^{1/2}_\nu} \Big)$$

where the constant $C(\|f\|_{\mathcal{F}^{1,1}}) \to 0$ as $\|f\|_{\mathcal{F}^{1,1}_\nu} \to 0$.

Summarizing,

$$\frac{1}{2}\frac{d}{dt}\|f\|_{L^2_\nu}^2(t) \leq \left(\nu - \frac{A_\rho}{4\pi} + c(\epsilon, \|f\|_{\mathcal{F}^{1,1}_\nu})\right)\|f\|^2_{\dot{H}^{1/2}_\nu} + \frac{1}{2\epsilon}\tilde{c}(\|f\|_{\mathcal{F}^{1,1}_\nu})\|f\|^2_{\mathcal{F}^{3/2,1}_\nu}\|f\|^2_{L^2_\nu}$$

where the constants go to 0 as $\|f\|_{\mathcal{F}^{1,1}_\nu} \to 0$ or as $\epsilon \to 0$.

Summarizing,

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_\nu}(t) \leq \left(\nu - \frac{A_\rho}{4\pi} + c(\epsilon, \|f\|_{\mathcal{F}^{1,1}_\nu})\right)\|f\|^2_{\dot{H}^{1/2}_\nu} + \frac{1}{2\epsilon}\tilde{c}(\|f\|_{\mathcal{F}^{1,1}_\nu})\|f\|^2_{\mathcal{F}^{3/2,1}_\nu}\|f\|^2_{L^2_\nu}$$

where the constants go to 0 as $\|f\|_{\mathcal{F}^{1,1}_\nu} \to 0$ or as $\epsilon \to 0$. For ϵ sufficiently small, by Gronwall's inequality,

$$\|f\|_{L^2_\nu}(t) \leq C\|f_0\|_{L^2} \exp\left(C\int_0^t \|f\|^2_{\mathcal{F}^{3/2,1}_\nu} dt\right)$$

Summarizing,

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_\nu}(t) \le \left(\nu - \frac{A_\rho}{4\pi} + c(\epsilon, \|f\|_{\mathcal{F}^{1,1}_\nu})\right)\|f\|^2_{\dot{H}^{1/2}_\nu} + \frac{1}{2\epsilon}\tilde{c}(\|f\|_{\mathcal{F}^{1,1}_\nu})\|f\|^2_{\mathcal{F}^{3/2,1}_\nu}\|f\|^2_{L^2_\nu}$$

where the constants go to 0 as $\|f\|_{\mathcal{F}^{1,1}_\nu} \to 0$ or as $\epsilon \to 0$. For ϵ sufficiently small, by Gronwall's inequality,

$$\|f\|_{L^2_\nu}(t) \le C\|f_0\|_{L^2} \exp\left(C\int_0^t \|f\|^2_{\mathcal{F}^{3/2,1}_\nu} dt\right)$$

Finally, the exponential term on the right hand side is uniformly bounded using interpolation

$$\int_0^t \|f\|^2_{\mathcal{F}^{3/2,1}_\nu} dt \le \int_0^t \|f\|_{\mathcal{F}^{1,1}_\nu}\|f\|_{\mathcal{F}^{2,1}_\nu} dt \le \|f\|_{L^\infty_t \mathcal{F}^{1,1}_\nu}\int_0^t \|f\|_{\mathcal{F}^{2,1}_\nu} dt \le \|f_0\|^2_{\mathcal{F}^{1,1}}.$$

Ill-posedness

The gain of Sobolev regularity motivates the ill-posedness of the unstable case $\rho_1 > \rho_2$.

Ill-posedness

The gain of Sobolev regularity motivates the ill-posedness of the unstable case $\rho_1 > \rho_2$. There exists initial data satisfying

$$\|f_0\|_{L^2} < \infty, \|f_0\|_{\mathcal{F}^{1,1}} < k_\mu \text{ and } \|f_0\|_{H^s} = \infty$$

for constant k_μ and $s > 0$,

Ill-posedness

The gain of Sobolev regularity motivates the ill-posedness of the unstable case $\rho_1 > \rho_2$. There exists initial data satisfying

$$\|f_0\|_{L^2} < \infty, \|f_0\|_{\mathcal{F}^{1,1}} < k_\mu \text{ and } \|f_0\|_{H^s} = \infty$$

for constant k_μ and $s > 0$, for example:
Let for $n \geq N$ for some $N > 0$ integer

$$\xi \hat{f}_0(\xi) = \begin{cases} n^\sigma & \text{if } \xi \in [n^\delta, n^\delta + 1/n^\gamma] \\ 0 & \text{otherwise} \end{cases}$$

such that $\gamma > \sigma + 1$, $2\delta + \gamma > 2\sigma + 1$ but $2\delta(1 - s) + \gamma = 2\sigma + 1$.

Remark

This example can be adapted to show that even if $f \in \mathcal{F}^{1,1}_\nu \cap L^2$, it need not be in H^s.

Ill-posedness

Theorem (Ill-posedness)

For every $s > 0$ *and* $\epsilon > 0$*, there exist a solution* $\tilde{f}$ *to the unstable regime and* $0 < \delta < \epsilon$ *such that* $\|\tilde{f}\|_{H^s}(0) < \epsilon$ *but* $\|\tilde{f}\|_{H^r}(\delta) = \infty$ *for any* $r > 0$*.*

This is significant because we show instantaneous blow-up of solutions in very low regularity spaces. In particular, one could start in H^s with high s and it still blows up in H^r for any small r.

Ill-posedness proof

Take $f_0 \in L^2 \cap \mathcal{F}^{1,1}$ for the Muskat problem in the stable regime such that $\|f_0\|_{H^r} = \infty$.

Ill-posedness proof

Take $f_0 \in L^2 \cap \mathcal{F}^{1,1}$ for the Muskat problem in the stable regime such that $\|f_0\|_{H^r} = \infty$. By the gain of regularity

$$\|f\|_{H^s}(\delta) \leq \|e^{-\nu\delta|\xi|}|\xi|^s\|_{L^\infty}\|f\|_{L^2_\nu}(\delta) \leq c(\delta)\|f_0\|_{L^2}\exp\left(\|f_0\|^2_{\mathcal{F}^{1,1}}\right) < \epsilon$$

by picking initial data with $\|f_0\|_{L^2} \ll 1$.

Ill-posedness proof

Take $f_0 \in L^2 \cap \mathcal{F}^{1,1}$ for the Muskat problem in the stable regime such that $\|f_0\|_{H^r} = \infty$. By the gain of regularity

$$\|f\|_{H^s}(\delta) \leq \|e^{-\nu\delta|\xi|}|\xi|^s\|_{L^\infty}\|f\|_{L^2_\nu}(\delta) \leq c(\delta)\|f_0\|_{L^2}\exp\left(\|f_0\|^2_{\mathcal{F}^{1,1}}\right) < \epsilon$$

by picking initial data with $\|f_0\|_{L^2} \ll 1$.If $f(x,t)$ is a solution to the stable case problem, then $\tilde{f}(x,t) = f(x,-t+\delta)$ is a solution to the unstable case $\rho_1 > \rho_2$. We conclude

$$\|\tilde{f}\|_{H^s}(0) = \|f\|_{H^s}(\delta) < \epsilon \text{ and } \|\tilde{f}\|_{H^r}(\delta) = \|f_0\|_{H^r} = \infty.$$

Rayleigh-Taylor unstable Muskat bubbles

$$\rho_t + u \cdot \nabla \rho = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0.$$

We recall the classical Darcy's law (κ is the permeability)

$$\frac{1}{\kappa}\mu(x,t)u(x,t) = -\nabla p(x,t) - \rho(x,t)e_3,$$

The densities and viscosities of each fluid are given by

$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t). \end{cases}$$

Surface tension at the interface is taken into consideration through the Laplace-Young's formula

$$p_1(x) - p_2(x) = \sigma k(x), \qquad x \in \partial D(t), \tag{1}$$

where $k(x)$ denotes the curvature of the curve $\partial D(t)$, $\sigma > 0$ the surface tension coefficient and $p_1(x)$, $p_2(x)$ the limit of pressure at x from inside and outside, respectively.

Equations

Then the boundary is parametrized as

$$\partial D^j(t) = \{z(\alpha,t) :\ \alpha \in [-\pi,\pi]\}$$

We will study again the dynamics of the free boundary $\partial D(t)$.

Since the fluids are assumed immiscible, the interface is just advected by the normal velocity field

$$z_t(\alpha,t)\cdot(\partial_\alpha z(\alpha,t))^\perp = BR(z(\alpha,t))\cdot(\partial_\alpha z(\alpha,t))^\perp$$

Here *BR* is the Birkhoff-Rott integral

$$BR(z,\omega)(\alpha,t) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi}\frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\omega(\beta,t)d\beta.$$

Equations continued...

The vorticity in this formulation is

$$\omega(\alpha,t) = 2A_\mu \mathcal{D}(z,\omega)(\alpha,t) + 2A_\sigma \partial_\alpha k(z(\alpha,t)) - 2A_\rho \partial_\alpha z_2(\alpha,t).$$

where

$$\mathcal{D}(z,\omega)(\alpha,t) = -BR(z,\omega)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)$$
$$= \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{(z(\alpha,t)-z(\beta,t))\cdot\partial_\alpha z(\alpha,t)^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\omega(\beta,t)d\beta.$$

and

$$A_\mu = \frac{\mu_2-\mu_1}{\mu_2+\mu_1}, \qquad A_\sigma = \frac{\kappa\sigma}{\mu_2+\mu_2}, \qquad A_\rho = \frac{g\kappa(\rho_2-\rho_1)}{\mu_2+\mu_1},$$

also the curvature is given by

$$k(\alpha,t) = \frac{\partial_\alpha z(\alpha,t)^\perp\cdot\partial_\alpha^2 z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^3}.$$

From these equations we have a closed system of equations for the contour evolution system.

Equilibria that are star-shaped bubbles

We will consider gravity driven *star-shaped bubbles*. That is, the boundary of the domain $D(t)$ can be parametrized by

$$z(\alpha, t) = R(1 + f(\alpha, t))(\cos\alpha, \sin\alpha) + (0, c(t)),$$

where R is determined as the radius of a circle with the same volume, $V(t)$, as $D(t)$.

Equilibria that are star-shaped bubbles

We will consider gravity driven *star-shaped bubbles*. That is, the boundary of the domain $D(t)$ can be parametrized by

$$z(\alpha,t) = R(1+f(\alpha,t))(\cos\alpha, \sin\alpha) + (0, c(t)),$$

where R is determined as the radius of a circle with the same volume, $V(t)$, as $D(t)$.

Since this volume is constant in time $V(t) = V_0$, due to incompressibility, then $R = \sqrt{\frac{V_0}{\pi}}$. Thus, $f(\alpha,t) > -1$ can be thought of as a radial perturbation. To simplify notation we shall write $f(\alpha,t) = f(\alpha)$ when there is no danger of confusion.

Equilibria that are star-shaped bubbles

We will consider gravity driven *star-shaped bubbles*. That is, the boundary of the domain $D(t)$ can be parametrized by

$$z(\alpha, t) = R(1 + f(\alpha, t))(\cos\alpha, \sin\alpha) + (0, c(t)),$$

where R is determined as the radius of a circle with the same volume, $V(t)$, as $D(t)$.

Since this volume is constant in time $V(t) = V_0$, due to incompressibility, then $R = \sqrt{\frac{V_0}{\pi}}$. Thus, $f(\alpha, t) > -1$ can be thought of as a radial perturbation. To simplify notation we shall write $f(\alpha, t) = f(\alpha)$ when there is no danger of confusion.

...After a good amount of computation the equation requires that

$$c'(t) = \frac{A_\rho}{2\pi} \mathrm{PV} \int_{-\pi}^{\pi} \frac{\cos(\beta/2)\cos(\alpha - \beta)}{\sin(\beta/2)\sin\alpha} d\beta = A_\rho.$$

So that when $f = 0$, the gravity driven circle is a steady state.

Without Surface Tension, the interface problem is Rayleigh-Taylor stable if it satisfies the Rayleigh-Taylor condition:

$$\sigma(\alpha, t) = -(\nabla p^2(z(\alpha, t), t) - \nabla p^1(z(\alpha, t), t)) \cdot \partial_\alpha^\perp z(\alpha, t) > 0$$

Without Surface Tension, the interface problem is Rayleigh-Taylor stable if it satisfies the Rayleigh-Taylor condition:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t)) \cdot \partial_\alpha^\perp z(\alpha,t) > 0$$

From Darcy's law this can be written as

$$\sigma(\alpha,t) = \frac{\mu_1 - \mu_2}{\kappa} BR(z,\omega)(\alpha,t) \cdot \partial_\alpha^\perp z(\alpha,t) + g(\rho^2 - \rho^1)\partial_\alpha z_1(\alpha,t)$$

Here we can see the importance of the relative position between the denser and lighter fluid.

For a closed curve, sigma cannot be everywhere positive as the integral of sigma on a closed curve is zero, so it has to be negative on part of the curve.

Intuitively, if the liquid of the bubble is lighter than the surrounding fluid (bubble going up), in the upper half of the bubble the lighter fluid is below the denser one.

What we show is that when surface tension is added, the regularizing effects allows for global existence in this situation when you are close enough to a circle.

Theorem (Existence and Uniqueness in 2D(GGPS))

Let $f_0 \in \dot{\mathcal{F}}^{1,1} \cap L^2$ *satisfy the bound*

$$\|f_0\|_{\dot{\mathcal{F}}^{1,1}} < c$$

for a constant $c = c(|A_\mu|, A_\sigma, A_\rho)$.
Then there exists a global in time unique solution to with $f \in L^\infty(0, T; \dot{\mathcal{F}}^{1,1} \cap L^2) \cap L^1(0, T; \dot{\mathcal{F}}^{4,1})$ *such that* $f(\alpha, 0) = f_0(\alpha)$,

$$\|f\|_{L^2}(t) \leq \|f_0\|_{L^2},$$

and

$$\|f\|_{\dot{\mathcal{F}}^{1,1}}(t) + \sigma \int_0^t \|f\|_{\dot{\mathcal{F}}^{4,1}}(\tau) d\tau \leq \|f_0\|_{\dot{\mathcal{F}}^{1,1}},$$

We also have the exponential decay on $[-\pi, \pi]$ and we can show the gain of Analytic regularity.

The Equation for the interface

Eventually we obtain the equation for the interface....

$$
\begin{aligned}
\partial_t f(\alpha) = &-\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{f(\alpha)-f(\alpha-\beta)}{1+f(\alpha)}\frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{1+f(\alpha-\beta)}{1+f(\alpha)}k_2(\alpha-\beta)\frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3(1+f(\alpha))}\frac{1}{2\pi}\mathrm{PV}\int \frac{k_3(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+2A_\mu\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int \frac{\mathcal{D}(f,\tilde{\omega})(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&-\frac{2A_\rho}{R}\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int \frac{\partial_\alpha f(\alpha-\beta)\sin(\alpha-\beta)+(1+f(\alpha-\beta)\cos(\alpha-\beta))}{2\sin(\beta/2)}d\beta \\
&\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int (N(\alpha,\beta)-1)\frac{\tilde{\omega}(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{1/4\pi}{1+f(\alpha)}\mathrm{PV}\int_{-\pi}^{\pi}\frac{\partial_\alpha f(\alpha)(1+f(\alpha-\beta))}{(\Delta_\beta f(\alpha))^2+(1+f(\alpha))(1+f(\alpha-\beta))}\tilde{\omega}(\alpha-\beta)d\beta \\
&+\frac{A_\rho}{R(1+f(\alpha))}\Big(\partial_\alpha f(\alpha)\cos\alpha-(1+f(\alpha))\sin\alpha\Big).
\end{aligned}
$$

where

$$
\tilde{\omega}(\alpha) = 2A_\mu\mathcal{D}(f,\tilde{\omega})(\alpha)+\frac{2A_\sigma}{R^3}\partial_\alpha k(f(\alpha))-\frac{2A_\rho}{R}\Big(\partial_\alpha f(\alpha)\sin\alpha+(1+f(\alpha))\cos\alpha\Big).
$$

We just used the following notation

$$N(\alpha,\beta)=\frac{\partial_\alpha f(\alpha)\Delta_\beta f(\alpha)+(1+f(\alpha))(1+f(\alpha-\beta))\cos(\beta/2)}{(\Delta_\beta f(\alpha))^2+(1+f(\alpha))(1+f(\alpha-\beta))},$$

$$\tilde{\omega}(\alpha)=2A_\mu\mathcal{D}(f,\tilde{\omega})(\alpha)+\frac{2A_\sigma}{R^3}\partial_\alpha k(f(\alpha))-\frac{2A_\rho}{R}\Big(\partial_\alpha f(\alpha)\sin\alpha+(1+f(\alpha))\cos\alpha\Big).$$

and we split $\partial_\alpha k(f(\alpha))$ in three terms,

$$\partial_\alpha k(f(\alpha))=-\partial_\alpha^3 f(\alpha)(1+f(\alpha))+k_2(\alpha)(1+f(\alpha))\partial_\alpha^3 f(\alpha)+k_3(\alpha),$$

where

$$k_2(\alpha)=\left(1-\frac{1}{((\partial_\alpha f(\alpha))^2+(1+f(\alpha))^2)^{3/2}}\right),$$

$$\begin{aligned}k_3(\alpha)=&\frac{1}{((\partial_\alpha f(\alpha))^2+(1+f(\alpha))^2)^{5/2}}\Big(-3\partial_\alpha^2 f(\alpha)(\partial_\alpha f(\alpha))^3+3(\partial_\alpha^2 f(\alpha))^2\partial_\alpha f(\alpha)(1+f(\alpha))\\&+3\partial_\alpha^2 f(\alpha)\partial_\alpha f(\alpha)(1+f(\alpha))^2-4(\partial_\alpha f(\alpha))^3(1+f(\alpha))-(1+f(\alpha))^3\partial_\alpha f(\alpha)\Big)\end{aligned}$$

We use a "Pseudo Hilbert" transform

$$S(g)(\alpha) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{g(\alpha-\beta)}{2\sin(\beta/2)}d\beta,$$

We use a "Pseudo Hilbert" transform

$$S(g)(\alpha) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi}\frac{g(\alpha-\beta)}{2\sin(\beta/2)}d\beta,$$

then

$$\begin{aligned}\widehat{S(g)}(\alpha)(k) &= \hat{g}(k)\frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi}\frac{e^{-ik\beta}}{2\sin(\beta/2)}d\beta\\ &= \hat{g}(k)\frac{-i}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi}\frac{\sin(k\beta)}{2\sin(\beta/2)}d\beta\\ &= -i\,\mathrm{sign}(k)m(k)\hat{f},\end{aligned}$$

where

$$m(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(|k|\beta)}{2\sin(\beta/2)}d\beta = \frac{1}{2\pi}\sum_{j=1}^{|k|}(-1)^{j+1}\frac{8}{2j-1}.$$

We also use a "pseudo derivative"

$$\Delta_\beta f(\alpha) = \frac{f(\alpha) - f(\alpha - \beta)}{2\sin(\beta/2)},$$

We also use a "pseudo derivative"

$$\Delta_\beta f(\alpha) = \frac{f(\alpha) - f(\alpha - \beta)}{2\sin(\beta/2)},$$

$$\widehat{\Delta_\beta f}(\alpha) = \frac{1 - e^{-ik\beta}}{2\sin(\beta/2)}\hat{f}(\beta) = \tilde{m}(k,\beta)\hat{f}(k),$$

$$\begin{aligned}
\tilde{m}(k,\beta) &= \frac{1 - e^{-ik\beta/2}e^{-ik\beta/2}}{2\sin(\beta/2)} \\
&= \frac{e^{ik\beta/2} - e^{-ik\beta/2}}{2\sin(\beta/2)} e^{-ik\beta/2} \\
&= ik\frac{\sin(k\beta/2)}{k\sin(\beta/2)} e^{-ik\beta/2},
\end{aligned}$$

The Equation for the interface

$$
\begin{aligned}
\partial_t f(\alpha) = &-\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{f(\alpha)-f(\alpha-\beta)}{1+f(\alpha)}\frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{1+f(\alpha-\beta)}{1+f(\alpha)}k_2(\alpha-\beta)\frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3(1+f(\alpha))}\frac{1}{2\pi}\mathrm{PV}\int \frac{k_3(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+2A_\mu\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int \frac{\mathcal{D}(f,\tilde{\omega})(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&-\frac{2A_\rho}{R}\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int \frac{\partial_\alpha f(\alpha-\beta)\sin(\alpha-\beta)+(1+f(\alpha-\beta)\cos(\alpha-\beta))}{2\sin(\beta/2)}d\beta \\
&\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int (N(\alpha,\beta)-1)\frac{\tilde{\omega}(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{1/4\pi}{1+f(\alpha)}\mathrm{PV}\int_{-\pi}^{\pi} \frac{\partial_\alpha f(\alpha)(1+f(\alpha-\beta))}{(\Delta_\beta f(\alpha))^2+(1+f(\alpha))(1+f(\alpha-\beta))}\tilde{\omega}(\alpha-\beta)d\beta \\
&+\frac{A_\rho}{R(1+f(\alpha))}\Big(\partial_\alpha f(\alpha)\cos\alpha-(1+f(\alpha))\sin\alpha\Big).
\end{aligned}
$$

Poincaré Inequality for volume preserving Bubbles

The volume preservation means that

$$V_0 = \pi R^2 = V(t) = \frac{1}{2}\int_{-\pi}^{\pi} R^2(1+f(\alpha,t))^2 d\alpha$$

This implies

$$\int_{-\pi}^{\pi} f(\alpha,t)d\alpha = -\frac{1}{2}\int_{-\pi}^{\pi}(f(\alpha,t))^2 d\alpha.$$

Since $f(x,t)$ changes sign then there exists $c(t)$ such that

$$f(x,t) = \int_{c(t)}^{x} f'(\alpha,t)d\alpha,$$

From here we can prove that for solutions we have

$$\|f\|_{\mathcal{F}^{0,1}} \leq C\|f\|_{\mathcal{F}^{1,1}}$$

A look at one term

$$
\begin{aligned}
\partial_t f(\alpha) = &-\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{f(\alpha)-f(\alpha-\beta)}{1+f(\alpha)}\frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3}\frac{1}{2\pi}\mathrm{PV}\int \frac{1+f(\alpha-\beta)}{1+f(\alpha)}k_2(\alpha-\beta)\frac{\partial_\alpha^3 f(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{2A_\sigma}{R^3(1+f(\alpha))}\frac{1}{2\pi}\mathrm{PV}\int \frac{k_3(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+2A_\mu\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int \frac{\mathcal{D}(f,\tilde{\omega})(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&-\frac{2A_\rho}{R}\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int \frac{\partial_\alpha f(\alpha-\beta)\sin(\alpha-\beta)+(1+f(\alpha-\beta)\cos(\alpha-\beta))}{2\sin(\beta/2)}d\beta \\
&\frac{1/2\pi}{1+f(\alpha)}\mathrm{PV}\int (N(\alpha,\beta)-1)\frac{\tilde{\omega}(\alpha-\beta)}{2\sin(\beta/2)}d\beta \\
&+\frac{1/4\pi}{1+f(\alpha)}\mathrm{PV}\int_{-\pi}^{\pi}\frac{\partial_\alpha f(\alpha)(1+f(\alpha-\beta))}{(\Delta_\beta f(\alpha))^2+(1+f(\alpha))(1+f(\alpha-\beta))}\tilde{\omega}(\alpha-\beta)d\beta \\
&+\frac{A_\rho}{R(1+f(\alpha))}\Big(\partial_\alpha f(\alpha)\cos\alpha-(1+f(\alpha))\sin\alpha\Big).
\end{aligned}
$$

$$\hat{\mathcal{D}}_2(k) = \frac{1}{2\pi} \sum_{n,m,l \geq 0} (-1)^{n+m+l} b_{m,n} b_{l,n+1} *^m \hat{f}(k) * \sum_{k_1} \cdots \sum_{k_{2n+l+1}} \left(\prod_{j=0}^{2n} i(k_j - k_{j+1}) \hat{f}(k_j - k_{j+1}) \right.$$

$$\left. \prod_{j=2n+1}^{2n+l} \hat{f}(k_j - k_{j+1}) \hat{\omega}(k_{2n+l+1}) I(k, k_1, \ldots, k_{2n+l+1}) \right),$$

$$
\hat{\mathcal{D}}_2(k) = \frac{1}{2\pi} \sum_{n,m,l\geq 0} (-1)^{n+m+l} b_{m,n} b_{l,n+1} *^{m} \hat{f}(k) * \sum_{k_1} \cdots \sum_{k_{2n+l+1}} \left(\prod_{j=0}^{2n} i(k_j - k_{j+1}) \hat{f}(k_j - k_{j+1}) \right.
$$

$$
\left. \prod_{j=2n+1}^{2n+l} \hat{f}(k_j - k_{j+1}) \hat{\omega}(k_{2n+l+1}) I(k, k_1, \ldots, k_{2n+l+1}) \right),
$$

$$
I(k, k_1, \ldots, k_{2n+l+1}) = \mathrm{PV} \int_{-\pi}^{\pi} \frac{d\beta}{2\sin(\beta/2)} \prod_{j=0}^{2n} \frac{\sin((k_j - k_{k+1})\beta/2)}{(k_j - k_{j+1})\sin(\beta/2)} e^{-i(k_j - k_{j+1})\beta/2}
$$

$$
\prod_{j=2n+1}^{2n+l} e^{-i(k_j - k_{j+1})\beta} e^{-ek_{2n+l+1}\beta}
$$

$$
= \mathrm{PV} \int_{-\pi}^{\pi} \frac{\sin((k + k_{2n+1} - 2k_{2n+l+1})\beta/2)}{2\sin(\beta/2)} \prod_{j=0}^{2n} \frac{\sin((k_j - k_{j+1})\beta/2)}{(k_j - k_{j+1})\sin(\beta/2)} d\beta
$$

We would like to have a good bound for I.

$$I = \mathrm{PV} \int_{-\pi}^{\pi} \frac{\sin(k\beta/2)}{2\sin(\beta/2)} \prod_{j=0}^{n} \frac{\sin(k_j\beta/2)}{k_j \sin(\beta/2)} d\beta$$

$$\frac{\sin(k_j\beta/2)}{\sin(\beta/2)} = \frac{e^{ik_j\beta/2} - e^{-ik_j\beta/2}}{e^{i\beta/2} - e^{-i\beta/2}} = \frac{e^{ik_j\beta/2(1-e^{-ik_j\beta})}}{e^{i\beta/2}(1-e^{-i\beta})}$$
$$= e^{i(k_j-1)\beta/2} \sum_{m=0}^{k_j-1} e^{-i\beta m} = \sum_{m=0}^{k_j-1} e^{i(-2m+k_j-1)\beta/2}.$$

$I(0,0) = 0$ and $I(1,0) = \pi$, and we can show that

$$I(k,0) = \begin{cases} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{2j-1} & \text{if } k = 2l, \\ \pi & \text{if } k = 2l+1, \end{cases}$$

In general

$$I = \mathrm{PV} \int_{-\pi}^{\pi} \frac{\sin(k\beta/2)}{2\sin(\beta/2)} \prod_{j=0}^{n} \frac{\sin(k_j\beta/2)}{k_j \sin(\beta/2)} d\beta$$

Eventually

$$I(A,k) = \frac{1}{2} \sum_{n=0}^{A-1} \frac{\sin\left((A-k-2n-1)\frac{\pi}{2}\right)}{A-k-2n-1}$$

$$I(k,A) = \frac{1}{2} \sum_{n=0}^{k-1} \frac{\sin\left((k-A-2n-1)\frac{\pi}{2}\right)}{k-A-2n-1}$$

In general

$$I = \text{PV} \int_{-\pi}^{\pi} \frac{\sin(k\beta/2)}{2\sin(\beta/2)} \prod_{j=0}^{n} \frac{\sin(k_j\beta/2)}{k_j \sin(\beta/2)} d\beta$$

Eventually

$$I(A,k) = \frac{1}{2} \sum_{n=0}^{A-1} \frac{\sin\left((A-k-2n-1)\frac{\pi}{2}\right)}{A-k-2n-1}$$

$$I(k,A) = \frac{1}{2} \sum_{n=0}^{k-1} \frac{\sin\left((k-A-2n-1)\frac{\pi}{2}\right)}{k-A-2n-1}$$

And finally

$$I(k,A) = \begin{cases} \frac{\pi}{4} & \text{if } k-A \text{ is odd,} \\ \frac{1}{2} \sum_{n=0}^{k-1} \frac{(-1)^{l-n+1}}{2(l-n)-1} & \text{if } k-A \text{ is even.} \end{cases}$$

So we conclude that $|I| \leq |I(k,A)| \leq \pi$ for all $0 \leq k, A \in \mathbb{Z}$.

THANK YOU!