Conditional regularity for the inhomogeneous Landau equation

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Non Standard Diffusions in Fluids, Kinetic Equations and Probability
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Introduction

The Landau equation (1936) models the evolution of a particle density $f(t, x, v) \geq 0$, where $t \geq 0$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$. It reads

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transport collision

where, for $\gamma \in [-3,1]$,

$$Q_L(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v-w)[f(w)\nabla_v f(v) - f(v)\nabla_v f(w)]\,\mathrm{d}w \right),$$

$$a(z) := c|z|^{\gamma+2}\left(\mathrm{Id} - \frac{z}{|z|} \otimes \frac{z}{|z|}\right).$$

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The collision operator Q_L is the limit of the non-cutoff Boltzmann collision operator Q_B as grazing collisions predominate (when $\gamma > -3$).

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- In plasma physics, collisions between charged particles are predominantly grazing in typical regimes. (This is modeled by the Coulomb ($\gamma = -3$) case of Landau.)
- Grazing collisions in the Boltzmann equation are a source of mathematical difficulties (cf. Grad's cutoff assumption) but also lead to smoothing effects.
- Besides its importance in plasma physics, the Landau equation can help us understand the role of grazing collisions in non-cutoff Boltzmann.

Introduction

The collision term can be written in divergence or nondivergence form:

$$\begin{aligned}\partial_t f + v\cdot\nabla_x f &= \nabla_v\cdot(\bar a^f(t,x,v)\nabla_v f) + \bar b^f(t,x,v)\cdot\nabla_v f + \bar c^f(t,x,v) f,\\ &= \mathrm{tr}(\bar a^f(t,x,v) D_v^2 f) + \bar c^f(t,x,v) f,\end{aligned}$$

where

$$\begin{aligned}\bar a^f(t,x,v) &:= a_\gamma \int_{\mathbb{R}^3}\left(Id - \frac{w}{|w|}\otimes\frac{w}{|w|}\right)|w|^{\gamma+2} f(t,x,v-w)\,\mathrm{d}w,\\ \bar b^f(t,x,v) &:= b_\gamma\int_{\mathbb{R}^3}|w|^\gamma w\, f(t,x,v-w)\,\mathrm{d}w,\\ \bar c^f(t,x,v) &:= \begin{cases} c_\gamma \int_{\mathbb{R}^3}|w|^\gamma f(t,x,v-w)\,\mathrm{d}w, & \gamma>-3,\\ cf(t,x,v), & \gamma=-3.\end{cases}\end{aligned}$$

Introduction

Cases:

- $\gamma = -3$: Coulomb potentials.
- $\gamma \in [-3, -2]$: very soft potentials.
- $\gamma \in (-2, 0)$: moderately soft potentials.
- $\gamma = 0$: Maxwellian molecules.
- $\gamma \in (0, 1]$: hard potentials.

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- The total mass, momentum, and energy are formally conserved:

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- Steady states: $f(t,x,v) = ce^{-\alpha|v-v_0|^2}$, called *Maxwellians*.

Well-posedness theory

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In the hydrodynamic scaling limit, the densities $\int f(t,x,v)\,\mathrm{d}v$, $\int v\,f(t,x,v)\,\mathrm{d}v$, and $\int |v|^2 f(t,x,v)\,\mathrm{d}v$ converge formally to a solution of the compressible Euler system (see e.g. Golse-Levermore, Golse-Saint-Raymond), which can develop singularities.

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Large-data global well-posedness is still unknown for $\gamma < -2$, even in the spatially homogeneous case!

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- Strategy: adapt De Giorgi's method to linear Fokker-Plank equations $\partial_t g + v \cdot \nabla_x g = \nabla_v \cdot (A \nabla_v g) + B \cdot \nabla_v g + s$ with $A, B, s \in L^\infty(Q_1)$.

Conditional regularity

In more detail: define

$$M_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v, \qquad \text{(mass density)}$$

$$E_f(t,x) = \int_{\mathbb{R}^3} |v|^2 f(t,x,v)\,\mathrm{d}v, \qquad \text{(energy density)}$$

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If $m_0 \le M_f(t,x) \le M_0$, $E_f(t,x) \le E_0$, and $H_f(t,x) \le H_0$, then the nonlocal coefficients $\bar{a}^f$, $\bar{b}^f$, $\bar{c}^f$ are bounded, $\bar{a}^f$ is uniformly elliptic.

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Therefore, the local Hölder estimates of [GIMV] for linear kinetic equations can be applied. Constants depend on m_0, M_0, E_0, H_0, and $\|f\|_{L^\infty}$.

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Cameron-Silvestre-S '17:

- Under the same assumptions as [GIMV], solutions satisfy global upper bounds of the form $f(t,x,v) \leq K(t^{-3/2}+1)(1+|v|)^{-1}$, for $\gamma \in (-2,0]$.

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- If, in addition, $f_{in} \leq Ce^{-\alpha|v|^2}$, then this decay is propagated. (Gaussian decay is not generated.)
- These constants depend on m_0, M_0, E_0, and H_0, but not on $\|f\|_{L^\infty}$.

Conditional regularity

Technical tools:

- Anisotropic upper and lower bounds for $\bar{a}^f$ of the form

$$\bar{a}^f_{ij}(t,x,v)e_i e_j \approx |e|^2 \begin{cases} (1+|v|)^\gamma, & e \parallel v, \\ (1+|v|)^{\gamma+2}, & e \perp v, \end{cases}$$

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  with constants depending on the mass, energy, and entropy.
- A change of variables that deals with this degenerating ellipticity.
- Comparison principle arguments with barriers of the form $e^{-\alpha|v|^2}$.

Conditional regularity

Theorem (Henderson-S '17)

If $\gamma \in (-2, 0]$ *and a weak solution* f *satisfies*

- $0 < m_0 \leq M_f(t,x) \leq M_0$, $E_f(t,x) \leq E_0$, $H_f(t,x) \leq H_0$ *for all* $t \in [0, T]$, $x \in \mathbb{R}^3$,
- $f_{in}(x,v) \leq Ce^{-\alpha|v|^2}$ *for some* $\alpha > 0$,

then $f \in C^\infty((0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$.

If $\gamma \in [-3, -2]$, *we must also assume an a priori bound on* $\|f\|_{L^\infty}$ *and* $\int |v|^p f(t,x,v)\,\mathrm{d}v$ *for* $p > 3/(5+\gamma)$.

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Any loss of smoothness can be detected at the macroscopic scale (when $\gamma \in (-2, 0]$).

Conditional regularity

Strategy: Prove Schauder estimates for linear Fokker-Plank equations

$$\partial_t g + v \cdot \nabla_x g = \mathrm{tr}(A D_v^2 g) + s,$$

with A, s Hölder continuous, and A uniformly elliptic.

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Properties:

- Diffusion acts only in v, but Hörmander's condition is satisfied.
- Symmetries: this class of equations is invariant under $(t, x, v) \mapsto (r^2 t, r^3 x, rv)$ and $(t, x, v) \mapsto (t_0 + t, x_0 + x + tv_0, v_0 + v)$.

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- Scaling suggests Schauder estimates would control g in $C_v^{2+\alpha} C_t^{1+\alpha/2} C_x^{(2+\alpha)/3}$, not enough to conclude weak solutions are classical.
- Even worse, Schauder actually only controls $\partial_t g + \nabla_x g$, not $\partial_t g$.

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To get around this difficulty, use coupling between solution f and coefficients $A = \bar{a}^f$, $s = \bar{c}^f f$, prove a second estimate with $A, s \in C^{1+\alpha}$.

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One finally has $f \in C^\alpha \Rightarrow f \in C^{3+\alpha}_{t,x,v}$. Differentiate equation (after applying change of variables) and bootstrap to obtain C^∞.

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For $\gamma \in [-3, -2]$, the constants depend additionally on $\|f\|_{L^\infty}$ and $P_0 = \sup_{t,x} \int |v|^p f(t,x,v) \, \mathrm{d}v$.

To get a good continuation criterion, we want to remove the dependence on some of these constants.

Conditional regularity

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Theorem (Henderson-S-Tarfulea '17)

If $\gamma \in (-2,0)$ and $e^{\alpha|v|^2} f_{in} \in H^4_{x,v}(\mathbb{R}^6)$, then a classical solution f exists on some interval $[0,T]$, with mass and energy densities uniformly bounded. If $f \not\equiv 0$, then f satisfies uniform pointwise lower bounds in any compact $K \subset (0,T] \times \mathbb{R}^6$, with constants depending on K, T, f_{in}, and the upper bounds on the mass and energy densities. In particular, $M_f(t,x)$ is uniformly positive for $t \geq t_0$, for any $t_0 > 0$.

For $\gamma \in [-3,-2]$, the same conclusion holds with constants depending additionally on $\|f\|_{L^\infty}$.

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Note that the initial data may contain vacuum regions.

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Define the stochastic process (X_t, V_t) by

$$\begin{cases} \mathrm{d}V_s = \bar{\sigma}_\varepsilon\left(t-s, X_s, V_s\right)\mathrm{d}W_s, \\ \mathrm{d}X_s = -V_s\,\mathrm{d}s, \\ V_0 = v, \quad X_0 = x. \end{cases}$$

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It can be shown that

$$f_\varepsilon(t,x,v) = \mathbb{E}\left[e^{\int_0^t \bar{c}^f(t-s,X_s,V_s)ds} f_{in}\,(X_t, V_t)\right],$$

where $f_\varepsilon \to f$ in $C^\alpha_{loc}([0,T] \times \mathbb{R}^6)$.

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- Step 3: Use transport term to spread mass from x_0 to any x, along trajectory $v \sim (x - x_0)/t$.

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- Step 1: Initial data has a positive "mass core" that is spread forward to small values of t.
- Step 2: The core of mass implies the diffusion matrix $\bar{\sigma}_\varepsilon$ is uniformly elliptic near $(x_0, v_0) \Rightarrow$ mass is spread to every velocity, near x_0.
- Step 3: Use transport term to spread mass from x_0 to any x, along trajectory $v \sim (x - x_0)/t$.
- Step 4: Repeat Step 2 to show mass is spread to every v at every x.

Conditional regularity

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In fact, this super-Gaussian rate of decay is achieved for certain initial data.

Conditional regularity

By combining these lower bounds with our previous *a priori* estimates, it follows that (for well-distributed data) the constructed solution is C^∞ and can be extended past $T > 0$ as long as

$$\begin{cases} \sup\limits_{\substack{0\le t\le T\\ x\in\mathbb{R}^3}} \left(M_f(t,x) + E_f(t,x)\right) < \infty, & \gamma \in (-2,0), \\ \sup\limits_{\substack{0\le t\le T\\ x\in\mathbb{R}^3}} \left(M_f(t,x) + E_f(t,x) + \|f(t,x,\cdot)\|_{L^\infty}\right), & \gamma \in [-3,-2]. \end{cases}$$

Conditional regularity

Theorem (S '18)

If $\gamma \in (0,1]$, and a weak solution f satisfies $0 < m_0 \le M_f(t,x) \le M_0, E_f(t,x) \le E_0, H_f(t,x) \le H_0$ and $\int_{\mathbb{R}^3} |v|^{\gamma+2} f \, \mathrm{d}v \le G_0$ for all $t \in [0,T]$, $x \in \mathbb{R}^3$, then for any $t_0 > 0$ we have

$$ce^{-\alpha|v|^2} \le f(t,x,v) \le Ce^{-\beta|v|^2}, \quad t \ge t_0,$$

where c, C, α, β depend on m_0, M_0, E_0, H_0, and t_0. (Not on G_0.)

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No decay assumption is needed on initial data, unlike in $\gamma \le 0$ case.

Looking forward

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1. Guaranteeing that mass and energy densities of f cannot concentrate at any x. (Not a problem in the space homogeneous case.)
2. For fixed x, guaranteeing the solution f cannot concentrate at any v. (Not a problem when $\gamma > -2$.)

See also:

There is a parallel program on conditional regularity for the non-cutoff Boltzmann equation, under similar *a priori* assumptions (bounds on the mass/energy/entropy):

- Silvestre CMP '16: local boundedness
- Imbert-Silvestre JEMS '18: Hölder regularity
- Imbert-Mouhot-Silvestre, preprint: decay for large $|v|$

Thank you!