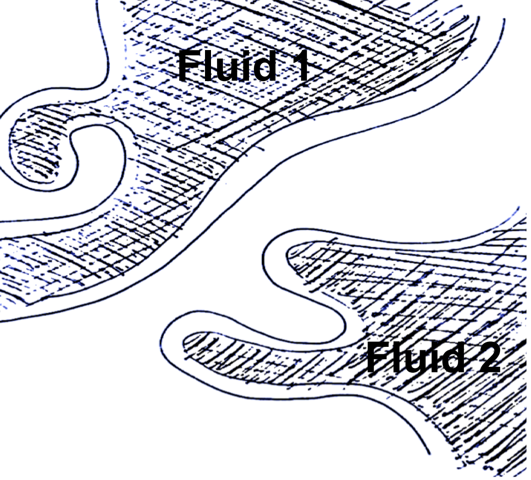
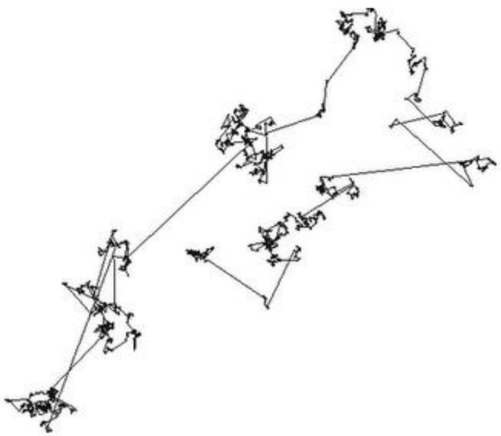
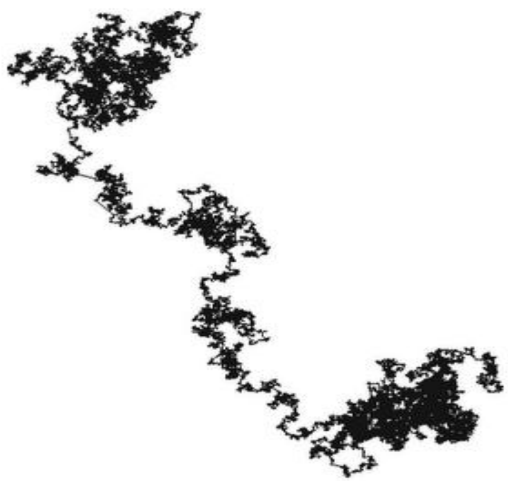






STABLE PHASE TRANSITIONS: FROM NONLOCAL TO LOCAL?

Joint works with A. Figalli X. Cabré, E. Cinti, S. Dipierro, E. Valdinoci

Non Standard Diffusions in Fluids, Kinetic Equations and Probability
(Marseille 10-14th December 2018)

Joaquim Serra

ETH zürich

The talked is based on

Cinti, S, Valdinoci,
Quantitative flatness results and BV-estimates for nonlocal minimal surfaces,
to appear in J. Diff. Geom.

Dipierro, S, Valdinoci,
Improvement of flatness for nonlocal phase transitions,
to appear in Amer. J. Math.

Cabré, Cinti, S,
Stable s-minimal cones in $\mathbb{R}^3$ *are flat for s*~1
preprint arXiv (2017)

Figalli, S,
On stable solutions for boundary reactions: a De Giorgi type result in dimension 4+1,
preprint arXiv (2017).

Cabré, Cinti, S,
Flatness of stable nonlocal phase transitions in $\mathbb{R}^3$
Forthcoming preprint (2018)

Outline of the talk

1. **Two models for phase transitions and crystal dislocations vs. minimal surfaces**
2. Stable solutions: known results and open questions
3. Nonlocal models and stronger consequences of stability
4. The De Giorgi conjecture for boundary reactions (or the half-Laplacian)
5. From nonlocal to local?

1. Two models for phase transitions and crystal dislocations vs. minimal surfaces

ALLEN-CAHN EQ'N (1950's)

Model for the **interphase** between **two fluids** / metal alloys / …

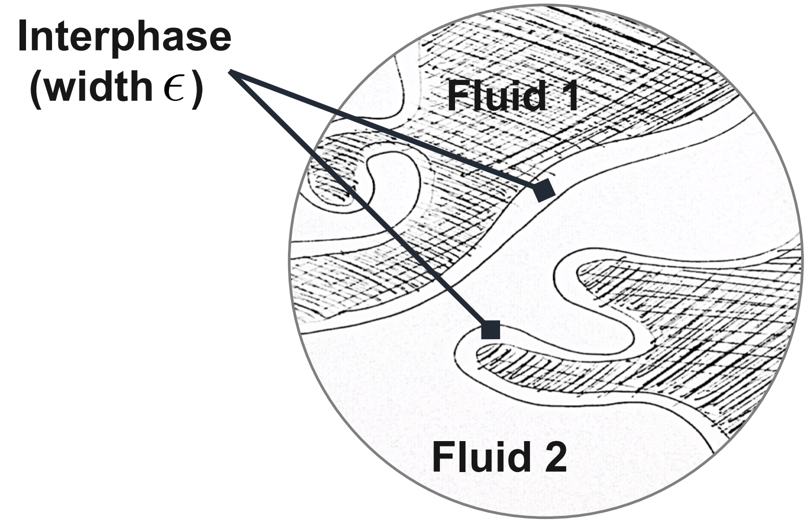
$$u : \mathbb{R}^n \to (-1,1)$$

Fluid 1: $u \sim +1$

Fluid 2: $u \sim -1$

$$(-\Delta)u + \epsilon^{-2}W'(u) = 0$$

$$\epsilon^2 \int_{\mathbb{R}^n} |\nabla u|^2 + W(u)\, dx$$

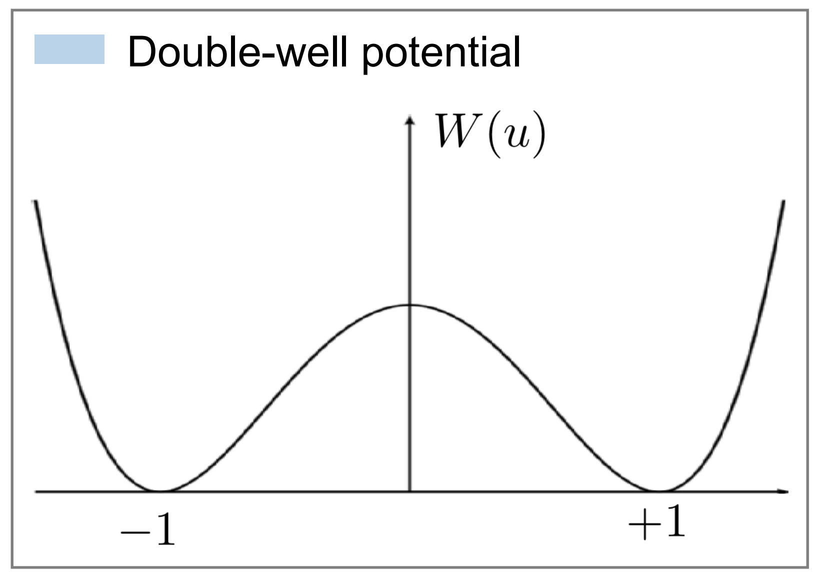


PEIERLS-NABARRO EQ'N (1940's)

Model for **crystal dislocations** / phase transition with line tension effect / …

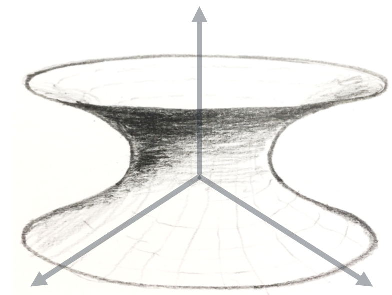
$$(-\Delta)^{1/2}u + \epsilon^{-1}W'(u) = 0$$

$$\epsilon \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+1}}\, dxdy + \int_{\mathbb{R}^n} W(u)\, dx$$



1. Two models for phase transitions and crystal dislocations vs. minimal surfaces

ALLEN-CAHN
PEIERLS-NABARRO } $\xrightarrow{\epsilon \downarrow 0}$ **MINIMAL SURFACES**



Let

$$I_\epsilon(u) := \epsilon^2 \int_{\mathbb{R}^n} |\nabla u|^2 + W(u)\, dx$$

$$J_\epsilon(u) := \epsilon \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+1}}\, dx dy + \int_{\mathbb{R}^n} W(u)\, dx$$

Theorem

u_{ϵ_k} *minimizers of either* I_{ϵ_k} *or* J_{ϵ_k} ⇒

- $u_{\epsilon_k} \xrightarrow{L^1_{\text{loc}}} \chi_E - \chi_{\mathbb{R}^n \setminus E}$ *as* $\epsilon_k \downarrow 0$
- $E \subset \mathbb{R}^n$ *is a* ***minimizer of perimeter***
- ∂E *is a minimal surface*

- Modica, Mortola 1977
- Alberti, Bouchitte, Seppecher 1998

Remark: Although *J* is nonlocal, the asymptotic limit is (as for *I*) the **local** perimeter!

Detailed in the next slide

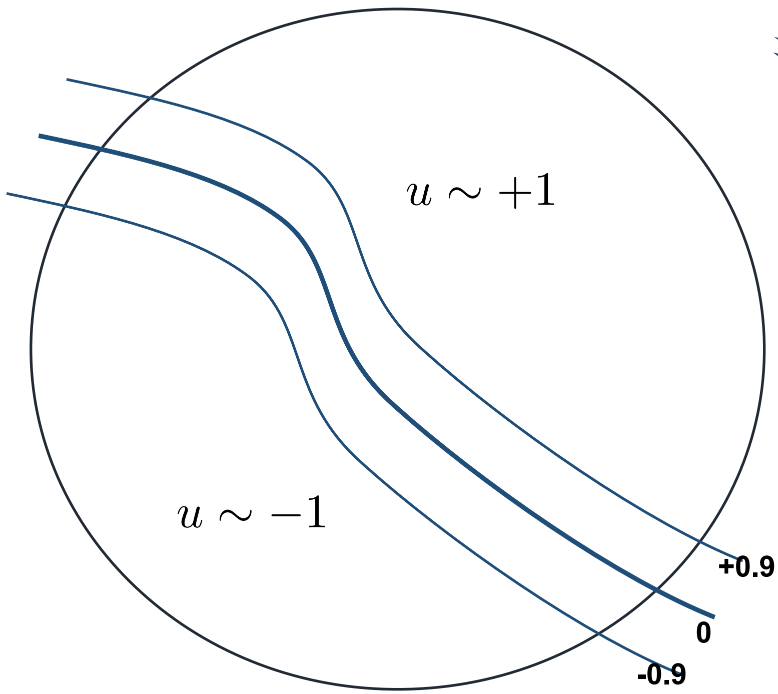
1. Two models for phase transitions and crystal dislocations vs. minimal surfaces

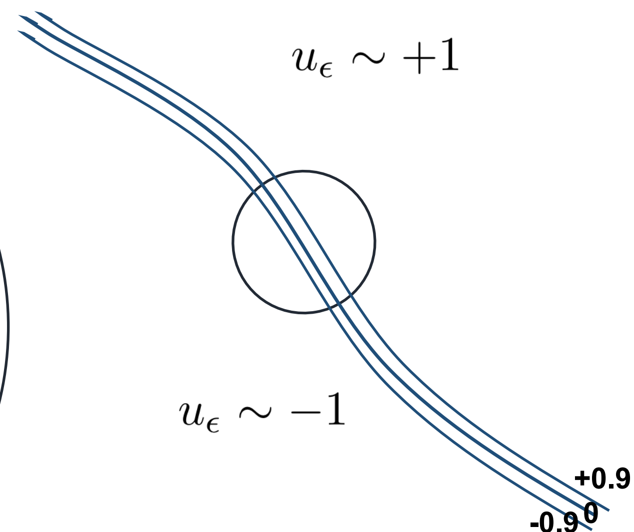
CONVERGENCE OF LEVEL SETS OF BLOW-DOWN SEQUENCES TO MININAL SURFACES

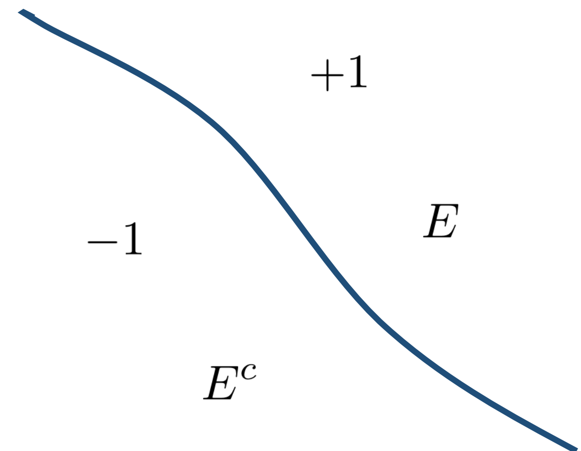
u *minimizer of* $\int |\nabla u|^2 + W(u)$

$u_\epsilon(x) := u(\epsilon x)$

$u_{\epsilon_k} \xrightarrow{L^1_{\mathrm{loc}}} \chi_E - \chi_{\mathbb{R}^n \setminus E}$



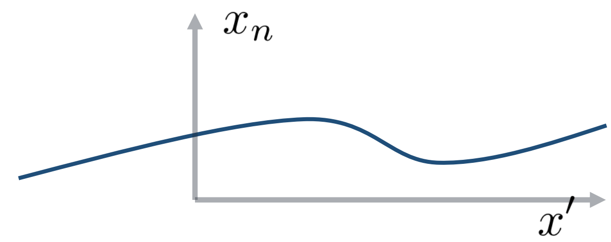




$E \subset \mathbb{R}^n$

minimizer of
perimeter!

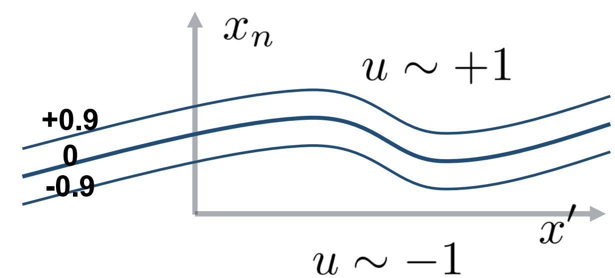
1. Two models for phase transitions and crystal dislocations vs. minimal surfaces



BERNSTEIN PROBLEM (1914) IN MINIMAL SURFACES …

Must any entire minimal graph[1] in $\mathbb{R}^n$ *be a hyperplane?*

- **Yes** for $n \leq 8$ (De Giorgi / Simons 1960's)
- **No** for $n \geq 9$ (Bombieri, De Giorgi, Giusti 1969)



… MOTIVATES A CONJECTURE OF DE GIORGI (1978)

$u : \mathbb{R}^n \to (-1,1)$ *sol'n of the Allen-Cahn eq'n with* $u_{x_n} > 0$

Must its level sets be hyperplanes if $n \leq 8$ *?*

- **Positive results:**
  - $n = 2$ Ghoussoub, Gui 1998;
  - $n = 3$ Ambrosio, Cabre 2000
  - $4 \leq n \leq 8$ Savin 2009 **under the extra assumption** $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$
- **(Counter)example for** $n \geq 9$**:** Del Pino, Kowalczyc, Wei 2011

1 Solutions to the minimal graph eq'n $\operatorname{div}\left(\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}\right) = 0 \qquad g : \mathbb{R}^{n-1} \to \mathbb{R}$

Outline of the talk

1. Two models for phase transitions and crystal dislocations vs. minimal surfaces

2. **Stable solutions: known results and open questions**

3. Nonlocal models and stronger consequences of stability

4. The De Giorgi conjecture for boundary reactions (or the half-Laplacian)

5. From nonlocal to local?

2. Stable solutions: known results and open questions

DEFINITION OF STABLE SOLUTIONS

- We say that a smooth, complete, **embedded minimal hypersurface** is **stable** if it is a stable critical point of the area functional

$$\Longleftrightarrow \qquad \int_S |\nabla\varphi|^2 - |I\!I_S|^2\varphi^2 \geq 0 \qquad \forall\varphi \in C_c^\infty(S)$$

- We say that a **sol'n of** the **Allen-Cahn** eq'n (resp. **Peierls-Nabarro** eq'n) is **stable** if it is a stable critical point of the energy functional

$$\Longleftrightarrow \qquad \int_{\mathbb{R}^n} |\nabla\varphi|^2 + W''(u)\varphi^2 \geq 0 \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^n)$$

STABILITY CONJECTURES

- Hyperplanes are the only smooth, complete, connected, embedded, stable minimal hypersurfaces in ambient dimensions $n \leq 7$

- Any stable solution of Allen-Cahn [resp. Peierls-Nabarro] is 1D symmetric in dimensions $n \leq 7$

Remark: Both statements are know to be true if we replace "Stable solutions" by "Energy minimizers"

2. Stable solutions: known results and open questions

MOTIVATIONS OF STABILITY CONJECTURES

- **Stability conjecture** for minimal surfaces is **~equivalent** to the following **universal curvature estimate**

$D_1 \subset \mathbb{R}^n$ *stable minimal disc* ⟹ $|I\!I| \le c(n)$ *in* $D_{1/2}$

- **Stability conjecture** for Allen-Cahn [resp. Peierls-Nabarro] is **~equivalent** to the following **universal curvature estimate** for level sets

$$u_\epsilon : B_1 \to (-1,1)$$

$$(-\Delta)u_\epsilon + \epsilon^{-2}W'(u_\epsilon) = 0 \quad \text{in} \quad B_1$$

$$\left[\ (-\Delta)^{1/2}u_\epsilon + \epsilon^{-1}W'(u_\epsilon) = 0\ \right]$$

⟹ $\left|I\!I_{\{u_\epsilon=0\}}\right| \le c(n)$ *in* $B_{1/2}$, $\forall \epsilon \le \epsilon_\circ(n)$

- **Stability conjecture** for minimal surfaces is **~equivalent** to **Schoen's conjecture** below

$D_1 \subset \mathbb{R}^n$ *stable minimal disc* ⟹ $\mathcal{H}^{n-1}(D_{1/2}) \le C(n)$

- **Stability conjecture** for Allen-Cahn [resp. Peierls-Nabarro] in dimension *n-1* **implies full* De Giorgi conjecture** in dimension *n*

Consider $U^{\pm}(x') := \lim_{x_n \to \pm\infty} u(x', x_n)$

* without assuming $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$

2. Stable solutions: known results and open questions

TABLE: STATE OF THE ART ON THE CONJECTURES

POSITIVE ANSWER
NEGATIVE ANSWER
OPEN QUESTION

	Energy **minimizers** are 1D symmetric?	**Stable** solutions satisfy area bounds? **~ Schoen conjecture**	**Stable** solutions are 1D symmetric? **Stability conjecture**
Allen-Cahn	**3D** (Ambrosio, Cabre 2000) **4D-8D** (Savin 2009) **From 9D** (Del Pino, Kowalczyk, Wei 2011)	**3D- ?**	**3D-7D ?** **8D-** (Pacard, Wei 2013)
Peierls-Nabarro	**3D** (Cabre, Cinti 2010) **4D-8D** (Savin 2018) **9D- ?**	**3D*** (Figalli, S 2017) **4D- ?**	**3D*** (Figalli, S 2017) **4D- ?**
Minimal surfaces	**3D-7D** (Simons 1968) **From 8D** (Bombieri, De Giorgi, Giusti 1969)	**3D** (Pogorelov 1981) **4D- ?**	**3D** (Do Carmo, Peng 1979; Fischer-Colbrie, Schoen 1980) **4D-7D ?** **8D-** (Simons 1968)

* Detailed later in part 4 of the talk

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3. **Nonlocal models and stronger consequences of stability**

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5. From nonlocal to local?

3. Nonlocal models and stronger consequences of stability

DEFINITION OF NONLOCAL PERIMETER

- Let L_K be the infinitesimal generator of a Levy process

$$L_K u(x) := \int_{\mathbb{R}^n} \big(u(x) - u(y)\big) K(x-y)dy \qquad K \geq 0$$

- Assume in addition

$$\int_{\mathbb{R}^n} \min(1, |z|) K(z) dz < +\infty$$

- Define the ***K*-perimeter** as

$$\mathrm{Per}_K(E) := \iint_{E^c \times E} K(x-y) dx dy \qquad E \subset \mathbb{R}^n$$

- We call it fractional ***s*-perimeter in** the particular case

$$K(z) = \frac{1}{|x-y|^{n+s}} \qquad 0 < s < 1$$

3. Nonlocal models and stronger consequences of stability

SOME MOTIVATIONS OF THE *K*-PERIMETER

- If we have a uniform density of particles each driven by Levy motion (L_K) then
  $\mathrm{Per}_K(E)$ measures the **intensity**[1] of **particles changing side** (between E and E^c)

- When one considers the **fractional Ginzburg-Landau energies**

$$E_{s,\epsilon}(u) := \epsilon^s \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big(u(x) - u(y)\big)^2}{|x-y|^{n+s}} dxdy + \int_{\mathbb{R}^n} W(u)dx \qquad s \in (0,2)$$

  interpolating **Peierls-Nabarro and Allen-Cahn** then the fractional *s*-perimeter arises in the limit $\epsilon \downarrow 0$ when $s < 1$

- Caffarelli, Souganidis 2010 "**cellular automats**" giving **motion by fractional mean curvature**

- Related with Bouet, Leonardi, Masnou 2016 definition of **mean curvature for point clouds**

- Fractional perimeters provided an elegant way to **estimate perimeters of pixeled images** (see Cinti, S, Valdinoci 2017)

1 Number of particles per second

3. Nonlocal models and stronger consequences of stability

SOME ANALOGIES BETWEEN CLASSICAL AND FRACTIONAL PERIMETERS

Links and analogies of fractional and classical perimeters	By	Year
▪ **Gamma-convergence** of the fractional perimeter to the classical perimeter as $s \to 1^-$	Ambrosio, De Philipis, Martinazzi	2011
▪ **Improvement of flatness** for minimizers of the nonlocal perimeter	Caffarelli, Roquejoffre, Savin	2010
▪ Flatness of s-**minimizing cones** in dimension **2**	Savin, Valdinoci	2013
▪ **Bernstein type result** in dimension **3**	Figalli, Valdinoci	2015
▪ Flatness of s-**minimizing cones** in dimensions **<8** for **s~1**	Caffarelli, Valdinoci	2015
▪ **Smoothness** of minimizers of the fractional perimeter in dimension **2** [resp. in dimensions **<8** for **s~1**]	Barrios, Figalli, Valdinoci	2012
▪ Co-area type formula relating $W^{s,1}$ norm of a function and the fractional perimeters of its level sets	Visintin	1991
▪ **Isoperimetric inequality** with deficit estimates	Figalli, Fusco, Maggi, Millot, Morini	2014
▪ Alexandroff type result for sets of **constant fractional mean curvature** and Delaunay like cylinders	Cabré, Fall, Solà-Morales, Weth	2015

3. Nonlocal models and stronger consequences of stability

DEFINITION OF STABLE SOLUTIONS

- We say that E with smooth boundary is **stable *K*-minimal** set if it is a stable critical point of P_K

$$\Longleftrightarrow \iint_{\partial E\times\partial E} \Big(\big(\varphi(x)-\varphi(y)\big)^2 - \big(\nu(x)-\nu(y)\big)^2 \varphi^2(x) \Big) K(x-y)\, d\mathcal{H}^{n-1}(x)\, d\mathcal{H}^{n-1}(y) \geq 0 \quad \forall \varphi \in C_c^\infty(\partial E)$$

- We say that a solution of the **fractional Allen-Cahn eq'n**

$$(-\Delta)^{s/2} u_\epsilon + \epsilon^{-s} W'(u_\epsilon) = 0 \qquad \textbf{(FAC)}$$

is **stable** if it is a stable critical point of the energy functional

$$\Longleftrightarrow \quad \epsilon^s \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(\varphi(x)-\varphi(y)\big)^2}{|x-y|^{n+s}}\, dx dy + \int_{\mathbb{R}^n} W''(u_\epsilon)\varphi^2\, dx \geq 0 \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n)$$

3. Nonlocal models and stronger consequences of stability

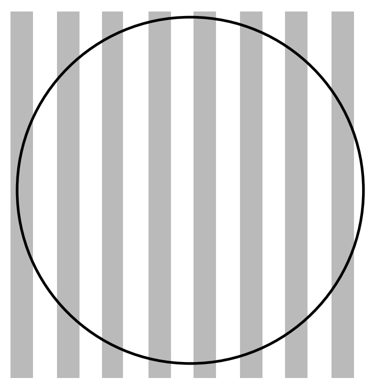
THE MORE NONLOCAL, THE STONGER IS THE INFORMATION GIVEN BY STABILITY

A striking difference with local perimeters and energy functionals: **new a priori BV estimates**

Cinti, S, Valdinoci 2017	$E \subset \mathbb{R}^n$ *stable s-minimal set in* B_1 *(with smooth boundary)*	➡	$\mathcal{H}^{n-1}(\partial E \cap B_{1/2}) \leq \frac{C(n)}{1-s}$
Cabre, Cinti, S 2018	$u_\epsilon : \mathbb{R}^n \to (-1,1)$ *solution of FAC, stable in* B_1 $s < 1$	➡	$\int_{B_{1/2}} \lvert\nabla u_\epsilon\rvert \leq \frac{C(n)}{1-s}$

Remark 1:
These estimates are in **every dimension**, and are a **nonlocal** (stronger) **analogue** of **Schoen's conjecture** (area bound for stable minimal surfaces)

Remark 2:
This picture is **not stable** in the ball for *s*-perimeter



Remark 3:

- BV estimate blows up as *s* to 1
- In particular, this does **not apply** to 1/2-Laplacian (*s*=1)
$$(-\Delta)^{1/2}u + \epsilon^{-1}W'(u) = 0$$
- Recall that the limit $\epsilon \downarrow 0$ of FAC gives local minimal surfaces for $1/2 \leq s/2 \leq 1$

3. Nonlocal models and stronger consequences of stability

USING THE NEW BV ESTIMATES WE CAN ADAPT THE CLASSICAL STRATEGY FOR MINIMIZING MINIMAL SURFACES …

1. Blow downs converge uniformly to stable cones

Cabre, Cinti, S 2018(2)

BV estimate

+

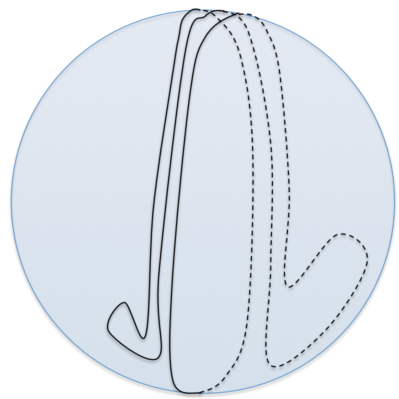
Control of Potential *energy* ***by Sobolev*** *energy*

+

Density estimates

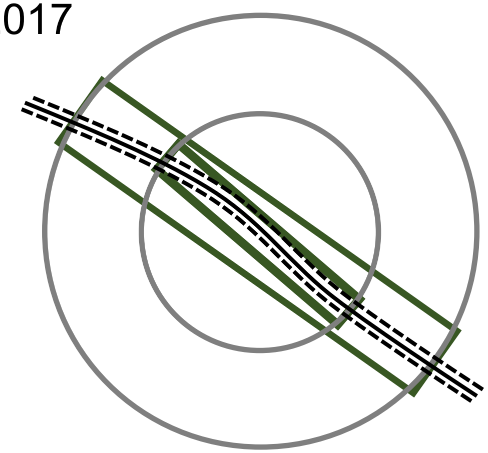
2. Classification of stable *s*-minimal cones

Cabre, Cinti, S 2018(1) only for ***n=3, s~1***



3. Improvement of flatness à la De Giorgi / Savin

Caffarelli, Roquejoffre, Savin 2010 / Diperro, S, Valdinoci 2017



… AND PROVE THE "ABSTRACT CLASSIFICTION RESULT" BELOW

n=3
s~1
Done!

Assume: for some pair (n,s) with 0<s<1 hyperplanes are the only stable s-minimal cones in dimension n

THEN → Cabre, Cinti, S 2018(2)

- *Hyperplanes are the only (smooth) minimal sets stable in the whole* $\mathbb{R}^n$
- *Every entire stable solution of FAC is 1D symmetric*

Outline of the talk

1. Two models for phase transitions and crystal dislocations vs. minimal surfaces

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4. **The De Giorgi conjecture for boundary reactions (or the half-Laplacian)**

5. From nonlocal to local?

4. The De Giorgi conjecture for boundary reactions (or the half-Laplacian)

FOR THE HALF LAPLACIAN THE NONLOCAL INFORMATION DEGENERATES AS $\epsilon \downarrow 0$ BUT, SURPRISINGLY, IT IS STILL ENOUGH TO PROVE THE STABILITY CONJECTUE IN DIMENSION 3

Figalli, S
2017

Theorem 1. Any **bounded stable** solution of any semilinear equation

$$(-\Delta)^{1/2}u = f(u) \quad \textit{in} \quad \mathbb{R}^3$$

is 1D symmetric and increasing (equivalently, its level sets are parallel planes)

Consequences:

- The stability conjecture for Peierls-Nabarro is true for *n*=3
- The "full" De Giorgi type result for Peierls-Nabarro is true for *n=4*

Theorem 2. In any dimension, a bounded any entire **stable** solution of Peierls-Nabarro ($\epsilon = 1$) satisfies the BV estimate

$$\int_{B_R} |\nabla u| \leq C(n, W) R^{n-1} \log R \qquad \forall R \geq 1$$

4. The De Giorgi conjecture for boundary reactions (or the half-Laplacian)

THE STRATEGY OF PROOF IS 100% "NONLOCAL STYLE"

Nonlocal approach

$$\int_{B_1} |\nabla u_\epsilon| dx \le C\left(1+\sqrt{\int_{B_2\times B_2} \frac{(u_\epsilon(x)-u_\epsilon(y))^2}{|x-y|^{n+1}}\,dxdy}\right)$$

+

A sharp interpolation inequality

$$w : B_2 \to (-1,1)$$

$$\int_{B_1\times B_1}\frac{(w(x)-w(y))^2}{|x-y|^{n+1}}\,dxdy \le C\log(1+\|\nabla w\|_{L^\infty})\left(1+\int_{B_2}|\nabla w|\,dx\right)$$

+

Scaling

New energy estimates

$$\tilde J(u_\epsilon,B_1) := \iint_{B_1\times B_1} \frac{(u_\epsilon(x)-u_\epsilon(y))^2}{|x-y|^{n+1}}\,dxdy \le C\log^2(1/\epsilon)$$

$$\tilde J(u,B_R) := \iint_{B_R\times B_R} \frac{(u(x)-u(y))^2}{|x-y|^{n+1}}\,dxdy \le C R^{n-1}\log^2 R$$

+

Log test function & multiscale recursion

$$n=3 \qquad \mathcal{A}(j) := \frac{\tilde J(u,B_{2^{2^j}})}{j2^{4j}}$$

$$\mathcal{A}(j) \le C\left(1+\sqrt{\frac{1}{2j}\sum_{k=1}^{2j}\mathcal{A}(j+k)}\right) \quad \Longrightarrow \quad \boxed{\mathcal{A}(j)\le C}$$

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5. From nonlocal to local?

Some open questions and future lines of research		Main difficulties
1)	Classification of stable *s*-minimal cones in dimension *n*=4 for *s*~1	▪ **No Simons** type **identity** (only with errors) ▪ Minimal surfaces in S^3 can be very complicated (compared with curves in S^2)
2)	Stability conjecture for FAC with 1<*s*<2 cones in dimension *n*=3	▪ The nonlocal proof does **not** give for 1<*s*<2 a **quasi-optimal area estimate** (as for *s*=1) ▪ **Monotonicity formula** is **not known** (it is only known for *s*=2 or for *s*<1)
3)	New *robust* proof of the area bound for stable minimal surfaces (*n*=3)	▪ No known replacement for **Gauss-Bonet** formula of surfaces

Thank you for your attention