Generalized Time Fractional Poisson Equations: Representations and Estimates

Kim, Panki (김 판기)

Seoul National University, Republic of Korea

Non standard diffusions in fluids, kinetic equations and probability,
CIRM, Marseille, France
December 10-14, 2018

References

Based on joints work with

Zhen-Qing Chen (University of Washington, USA)

Takashi Kumagai (RIMS, Kyoto University, Japan)

Jian Wang (Fujian Normal University, China).

- [CKKW] Zhen-Qing Chen, Panki Kim, Takashi Kumagai, Jian Wang, Time Fractional Poisson Equations: Representations and Estimates, Preprint
- [CKKW0] Zhen-Qing Chen, Panki Kim, Takashi Kumagai, Jian Wang, Heat kernel estimates for time fractional equations. *Forum Mathematicum*, 30(5), 1163-1192, (2018)

Outline

Outline

Time-fractional diffusion

The classical heat equation $\partial_t u = \Delta u$ describes heat propagation in homogeneous medium. The time-fractional diffusion equation $\partial_t^\beta u = \Delta u$ with $0 < \beta < 1$ has been widely used to model the anomalous diffusions exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena.

The (classical) fractional time derivative ∂_t^β in this talk is the Caputo derivative of order $\beta \in (0,1)$, which can be defined by

$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} (f(s) - f(0))\, ds.$$

Time-fractional diffusion

The classical heat equation $\partial_t u = \Delta u$ describes heat propagation in homogeneous medium. The time-fractional diffusion equation $\partial_t^\beta u = \Delta u$ with $0 < \beta < 1$ has been widely used to model the anomalous diffusions exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena.

The (classical) fractional time derivative ∂_t^β in this talk is the Caputo derivative of order $\beta \in (0,1)$, which can be defined by

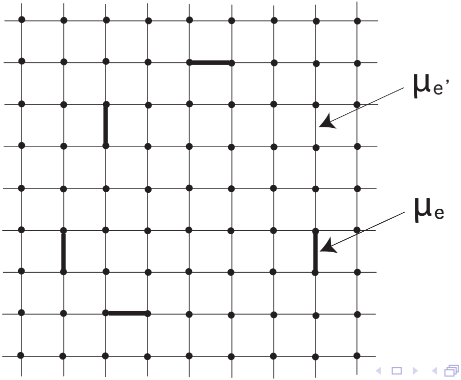
$$\partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} \left(f(s) - f(0)\right) ds.$$

Random conductance model

$\{\mu_e\}$: random conductance, *i.i.d.* on each edge e of $\mathbb{Z}^d$ s.t. $\exists \beta \in (0,1)$

$$\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\beta}(1 + o(1)) \text{ as } u \to \infty. \tag{1.1}$$

(Note that $\mathbb{E}\mu_e = \infty$.)
$\{X_t\}_{t\geq 0}$: cont. time Markov chain on $\mathbb{Z}^d$ (holding time $\exp(1)$).



Theorem 1.1 ($d \geq 3$ (Barlow-Černý '11))

$$\varepsilon X_{ct\varepsilon^{-2/\beta}} \to \mathbf{FK}_{d,\beta}(t) := BM_d([S_t^\beta]^{-1}) \quad \mathbf{P}\text{-a.s. on } D([0,\infty), \mathbb{R}^d),$$

where $\{S_t^\beta\}_{t\geq 0}$*:* β*-stable subordinator independent of Brownian motion* $\{BM_d(t)\}$*.*

$$\mathbb{E}[\exp\{-\lambda S_t^\beta\}] = \exp\{-t\lambda^\beta\}, \; [S_t^\beta]^{-1} = \inf\{s > 0 : S_\beta(s) > t\}.$$

Theorem 1.2 ($d = 2$ (Černý '11))

Same result by replacing $\varepsilon^{-2/\beta}$ *to* $\varepsilon^{-2/\beta}(\log \varepsilon^{-1})^{1-1/\beta}$*.*

Note that $BM_d(S_t^\beta)$ is isotropic 2β-stable process whose infinitesimal generator is $-(-\Delta)^\beta$, which gives "super-diffusive" behavior.

Theorem 1.1 ($d \geq 3$ (Barlow-Černý '11))

$$\varepsilon X_{ct\varepsilon^{-2/\beta}} \to \mathbf{FK}_{d,\beta}(t) := BM_d([S_t^\beta]^{-1}) \quad \mathbf{P}\text{-}a.s. \text{ on } D([0,\infty), \mathbb{R}^d),$$

where $\{S_t^\beta\}_{t\geq 0}$*:* β*-stable subordinator independent of Brownian motion* $\{BM_d(t)\}$*.*

$\mathbb{E}[\exp\{-\lambda S_t^\beta\}] = \exp\{-t\lambda^\beta\}$, $[S_t^\beta]^{-1} = \inf\{s > 0 : S_\beta(s) > t\}$.

Theorem 1.2 ($d = 2$ (Černý '11))

Same result by replacing $\varepsilon^{-2/\beta}$ *to* $\varepsilon^{-2/\beta}(\log \varepsilon^{-1})^{1-1/\beta}$*.*

Note that $BM_d(S_t^\beta)$ is isotropic 2β-stable process whose infinitesimal generator is $-(-\Delta)^\beta$, which gives "super-diffusive" behavior.

$\mathbf{FK}_{d,\beta}$: Fractional-kinetics process

— It is no longer a Markov process!

Density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation (fractional diffusion equation):

$$\partial_t^\beta p(t,x) = \frac{1}{2}\Delta p(t,x).$$

$\mathbf{FK}_{d,\beta}$: Fractional-kinetics process

— It is no longer a Markov process!

Density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation (fractional diffusion equation):

$$\partial_t^\beta p(t,x) = \frac{1}{2}\Delta p(t,x).$$

$\mathbf{FK}_{d,\beta}$: Fractional-kinetics process

— It is no longer a Markov process!

Density of its fixed time distribution $p(t,x)$ satisfies the fractional-kinetics equation (fractional diffusion equation):

$$\partial_t^\beta p(t,x) = \frac{1}{2}\Delta p(t,x).$$

Classical case: $0 < \beta < 1$

Riemann-Liouville fractional integral operator:

$$I^{\beta}\psi(t) = \Gamma(\beta)^{-1}\int_0^t (t-s)^{\beta-1}\psi(s)ds.$$

Caputo derivative:

$$\partial_t^{\beta}\psi(t) := \frac{d}{dt}I^{1-\beta}(\psi-\psi(0))(t) = \frac{d}{dt}I^{1-\beta}\psi(t) - \frac{\psi(0)}{t^{\beta}\Gamma(1-\beta)}.$$

Fractional diffusion equation in $\mathbb{R}^d$:

$$\partial_t^{\beta}p(t,x) = \Delta p(t,x) \qquad t>0, x\in\mathbb{R}^d.$$

Estimates of $p(t,x)$ were obtained (e.g. Eidelman-Kochubei ('04,)) by

$E_{\beta}(z) = \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}$: Mittag-Leffler function

$p(t,x) = \mathcal{F}^{-1}(E_{\beta}(|\xi|^2 t^{\beta}))$ and using Fourier analysis.

Classical case: $0 < \beta < 1$

Riemann-Liouville fractional integral operator:

$$I^\beta \psi(t) = \Gamma(\beta)^{-1} \int_0^t (t-s)^{\beta-1} \psi(s) ds.$$

Caputo derivative:

$$\partial_t^\beta \psi(t) := \frac{d}{dt} I^{1-\beta}(\psi - \psi(0))(t) = \frac{d}{dt} I^{1-\beta} \psi(t) - \frac{\psi(0)}{t^\beta \Gamma(1-\beta)}.$$

Fractional diffusion equation in $\mathbb{R}^d$:

$$\partial_t^\beta p(t,x) = \Delta p(t,x) \qquad t > 0, x \in \mathbb{R}^d.$$

Estimates of $p(t,x)$ were obtained (e.g. Eidelman-Kochubei ('04,)) by

$E_\beta(z) = \sum_{k=1}^\infty \frac{z^k}{\Gamma(\beta k + 1)}$: Mittag-Leffler function

$p(t,x) = \mathcal{F}^{-1}(E_\beta(|\xi|^2 t^\beta))$ and using Fourier analysis.

Classical case: $0 < \beta < 1$

Riemann-Liouville fractional integral operator:

$$I^{\beta}\psi(t) = \Gamma(\beta)^{-1}\int_0^t (t-s)^{\beta-1}\psi(s)ds.$$

Caputo derivative:

$$\partial_t^{\beta}\psi(t) := \frac{d}{dt}I^{1-\beta}(\psi-\psi(0))(t) = \frac{d}{dt}I^{1-\beta}\psi(t) - \frac{\psi(0)}{t^{\beta}\Gamma(1-\beta)}.$$

Fractional diffusion equation in $\mathbb{R}^d$:

$$\partial_t^{\beta}p(t,x) = \Delta p(t,x) \qquad t>0, x\in\mathbb{R}^d.$$

Estimates of $p(t,x)$ were obtained (e.g. Eidelman-Kochubei ('04,)) by

$E_{\beta}(z) = \sum_{k=1}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}$: Mittag-Leffler function

$p(t,x) = \mathcal{F}^{-1}(E_{\beta}(|\xi|^2 t^{\beta}))$ and using Fourier analysis.

(Q) More general case? (General spaces, general operators)

Motivation: Questions from industry

• The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.): Predict the progress of soil contamination.

• The third slide: Nature (2006, Jan.) by D. Brockmann, L. Hufnagel and T. Geisel: Human travel.

(Q) More general case? (General spaces, general operators)

Motivation: Questions from industry

• The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.): Predict the progress of soil contamination.

• The third slide: Nature (2006, Jan.) by D. Brockmann, L. Hufnagel and T. Geisel: Human travel.

(Q) More general case? (General spaces, general operators)

Motivation: Questions from industry

• The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.): Predict the progress of soil contamination.

• The third slide: Nature (2006, Jan.) by D. Brockmann, L. Hufnagel and T. Geisel: Human travel.

(Q) More general case? (General spaces, general operators)

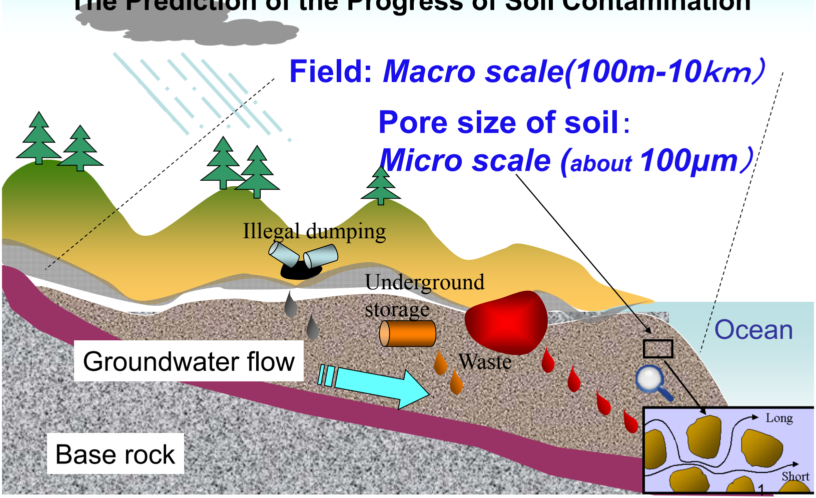
Motivation: Questions from industry

• The next two slides: J. Math. Ind. (2010) are by J. Nakagawa (Nippon Steel Co.): Predict the progress of soil contamination.

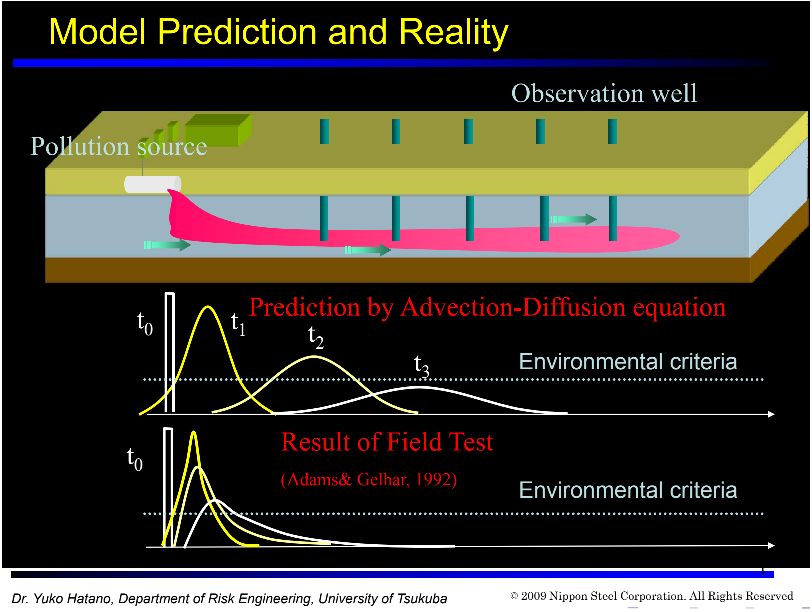
• The third slide: Nature (2006, Jan.) by D. Brockmann, L. Hufnagel and T. Geisel: Human travel.

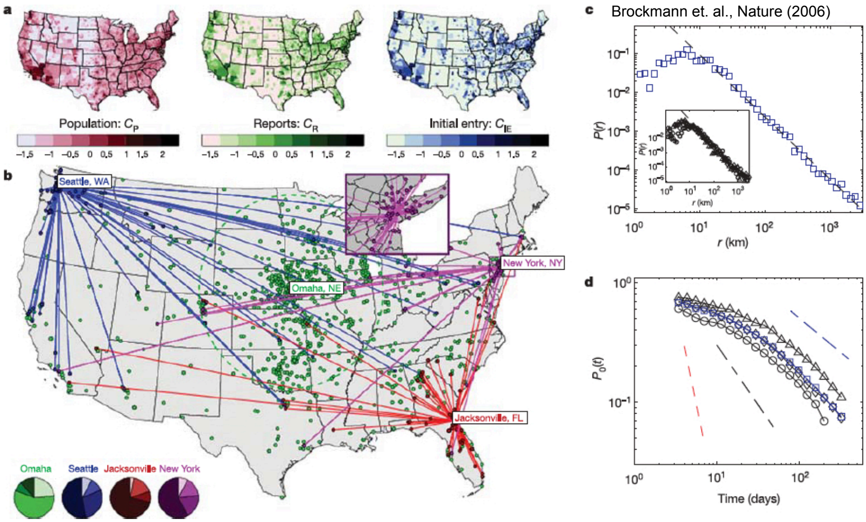
Issues Seen by Academia Engineering Researchers

“The Prediction of the Progress of Soil Contamination”



Dr. Yuko Hatano, Department of Risk Engineering, University of Tsukuba


Model Prediction and Reality
Observation well
Pollution source
Prediction by Advection-Diffusion equation
t0
t1
t2
t3
Environmental criteria
Result of Field Test
(Adams& Gelhar, 1992)
t0
Environmental criteria
Dr. Yuko Hatano, Department of Risk Engineering, University of Tsukuba
© 2009 Nippon Steel Corporation. All Rights Reserved


a
Population: C_P
Reports: C_R
Initial entry: C_{IE}
−1.5 −1 −0.5 0 0.5 1 1.5 2
b
Seattle, WA
Omaha, NE
New York, NY
Jacksonville, FL
Omaha
Seattle
Jacksonville
New York
c
Brockmann et. al., Nature (2006)
$P(r)$
r (km)
d
$P_0(t)$
Time (days)

Outline

Notation

Throughout this talk, we write $h(s) \simeq f(s)$, if there exist constants $c_1, c_2 > 0$ such that $c_1 f(s) \le h(s) \le c_2 f(s)$ for the specified range of the argument s.

Similarly, we write $h(s) \asymp f(s)g(s)$, if there exist constants $C_1, c_1, C_2, c_2 > 0$ such that $f(C_1 s)g(c_1 s) \le h(s) \le f(C_2 s)g(c_2 s)$ for the specified range of s.

For $a, b \in \mathbb{R}$ we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Notation

Throughout this talk, we write $h(s) \simeq f(s)$, if there exist constants $c_1, c_2 > 0$ such that $c_1 f(s) \le h(s) \le c_2 f(s)$ for the specified range of the argument s.

Similarly, we write $h(s) \asymp f(s)g(s)$, if there exist constants $C_1, c_1, C_2, c_2 > 0$ such that $f(C_1 s)g(c_1 s) \le h(s) \le f(C_2 s)g(c_2 s)$ for the specified range of s.

For $a, b \in \mathbb{R}$ we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Notation

Throughout this talk, we write $h(s) \simeq f(s)$, if there exist constants $c_1, c_2 > 0$ such that $c_1 f(s) \le h(s) \le c_2 f(s)$ for the specified range of the argument s.

Similarly, we write $h(s) \asymp f(s)g(s)$, if there exist constants $C_1, c_1, C_2, c_2 > 0$ such that $f(C_1 s)g(c_1 s) \le h(s) \le f(C_2 s)g(c_2 s)$ for the specified range of s.

For $a, b \in \mathbb{R}$ we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Subordinator

Throughout this talk, $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator ($S_t \geq 0$) with $S_0 = 0$ and ϕ is the Laplace exponent of S. That is,

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda > 0, t \geq 0.$$

The Laplace exponent ϕ of S is also called Bernstein function (vanishes at the origin) in the literature.

Since S has no drift, it is well known that there is a unique measure ν on $(0, \infty)$, which is called Lévy measure of S, satisfying $\int_0^\infty (1 \wedge x)\, \nu(dx) < \infty$ such that

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx).$$

We will assume that S_t has a bounded density $\bar{p}(t, \cdot)$ for each $t > 0$.

Let

$$w(x) := \nu(x, \infty)$$

and we always assume that the Lévy measure ν is infinite , which is equivalent to saying $w(x)$ is unbounded. ($w(x)$ is locally integrable on $[0, \infty)$)

Subordinator

Throughout this talk, $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator ($S_t \geq 0$) with $S_0 = 0$ and ϕ is the Laplace exponent of S. That is,

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda > 0, t \geq 0.$$

The Laplace exponent ϕ of S is also called Bernstein function (vanishes at the origin) in the literature.

Since S has no drift, it is well known that there is a unique measure ν on $(0, \infty)$, which is called Lévy measure of S, satisfying $\int_0^\infty (1 \wedge x)\, \nu(dx) < \infty$ such that

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx).$$

We will assume that S_t has a bounded density $\bar{p}(t, \cdot)$ for each $t > 0$.
Let

$$w(x) := \nu(x, \infty)$$

and we always assume that the Lévy measure ν is infinite , which is equivalent to saying $w(x)$ is unbounded. ($w(x)$ is locally integrable on $[0, \infty)$)

Subordinator

Throughout this talk, $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator ($S_t \geq 0$) with $S_0 = 0$ and ϕ is the Laplace exponent of S. That is,

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda > 0, t \geq 0.$$

The Laplace exponent ϕ of S is also called Bernstein function (vanishes at the origin) in the literature.

Since S has no drift, it is well known that there is a unique measure ν on $(0, \infty)$, which is called Lévy measure of S, satisfying $\int_0^\infty (1 \wedge x)\, \nu(dx) < \infty$ such that

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx).$$

We will assume that S_t has a bounded density $\bar{p}(t, \cdot)$ for each $t > 0$. Let

$$w(x) := \nu(x, \infty)$$

and we always assume that the Lévy measure ν is infinite , which is equivalent to saying $w(x)$ is unbounded. ($w(x)$ is locally integrable on $[0, \infty)$)

Subordinator

Throughout this talk, $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator ($S_t \geq 0$) with $S_0 = 0$ and ϕ is the Laplace exponent of S. That is,

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda > 0, t \geq 0.$$

The Laplace exponent ϕ of S is also called Bernstein function (vanishes at the origin) in the literature.

Since S has no drift, it is well known that there is a unique measure ν on $(0,\infty)$, which is called Lévy measure of S, satisfying $\int_0^\infty (1 \wedge x)\, \nu(dx) < \infty$ such that

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx).$$

We will assume that S_t has a bounded density $\bar{p}(t,\cdot)$ for each $t > 0$.

Let

$$w(x) := \nu(x,\infty)$$

and we always assume that the Lévy measure ν is infinite , which is equivalent to saying $w(x)$ is unbounded. ($w(x)$ is locally integrable on $[0,\infty)$)

Subordinator

Throughout this talk, $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator ($S_t \geq 0$) with $S_0 = 0$ and ϕ is the Laplace exponent of S. That is,

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \quad \lambda > 0, t \geq 0.$$

The Laplace exponent ϕ of S is also called Bernstein function (vanishes at the origin) in the literature.
Since S has no drift, it is well known that there is a unique measure ν on $(0, \infty)$, which is called Lévy measure of S, satisfying $\int_0^\infty (1 \wedge x)\, \nu(dx) < \infty$ such that

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, \nu(dx).$$

We will assume that S_t has a bounded density $\bar{p}(t, \cdot)$ for each $t > 0$.
Let

$$w(x) := \nu(x, \infty)$$

and we always assume that the Lévy measure ν is infinite , which is equivalent to saying $w(x)$ is unbounded. ($w(x)$ is locally integrable on $[0, \infty)$)

General time fractional derivative

Using $w(x) = \nu(x, \infty)$, the time fractional derivative with weight w is defined by

$$\partial_t^w f(t) := \frac{d}{dt} \int_0^t w(t-s)(f(s) - f(0))\, ds,$$

whenever the right hand side is well defined.
Define

$$I^w f(t) := \int_0^t w(t-s) f(s)\, ds.$$

Clearly, for any locally integrable function f on $[0, \infty)$

$$\partial_t^w f(t) = \frac{d}{dt} I^w f(t) - w(t) f(0) \quad \text{for a.e. } t > 0.$$

General time fractional derivative

Using $w(x) = \nu(x, \infty)$, the time fractional derivative with weight w is defined by

$$\partial_t^w f(t) := \frac{d}{dt} \int_0^t w(t-s)(f(s) - f(0))\, ds,$$

whenever the right hand side is well defined.
Define

$$I^w f(t) := \int_0^t w(t-s) f(s)\, ds.$$

Clearly, for any locally integrable function f on $[0, \infty)$

$$\partial_t^w f(t) = \frac{d}{dt} I^w f(t) - w(t) f(0) \quad \text{for a.e. } t > 0.$$

Theorem 1. Existence and uniqueness of Strong solution

Suppose that $\{P_t^0;\, t \geq 0\}$ is a uniformly bounded and strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ over a locally compact separable metric space E and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.

Let $T_0 \in (0,\infty)$, $g \in \mathcal{D}(\mathcal{L})$ and $f(t,x)$ be a function defined on $(0,T_0] \times E$ so that for a.e. $t \in (0,T_0]$, $f(t,\cdot) \in \mathcal{D}(\mathcal{L})$and

$$\int_0^{T_0} \|\mathcal{L} f(t,\cdot)\|\, dt < \infty \quad \text{and} \quad \|f(t,\cdot)\| \leq K < \infty \text{ for a.e. } t \in (0,T_0].$$

Then the function

$$\begin{aligned} u(t,x) &:= \mathbb{E}\left[P^0_{S_t^{-1}} g(x)\right] + \mathbb{E}\left[\int_{r=0}^{\infty} \mathbf{1}_{\{S_r<t\}} P_r^0 f(t-S_r,\cdot)(x)\, dr\right] \\ &= \mathbb{E}\left[P^0_{S_t^{-1}} g(x)\right] + \int_{s=0}^{t}\int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x)\bar{p}(r,t-s)\, dr\, ds \end{aligned} \tag{2.1}$$

is the unique strong solution of

$$\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x) \text{ on } (0,T_0] \times E \text{ with } u(0,x) = g(x)$$

in the following sense:

Theorem 1. Existence and uniqueness of Strong solution

Suppose that $\{P_t^0;\, t \geq 0\}$ is a uniformly bounded and strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ over a locally compact separable metric space E and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.
Let $T_0 \in (0, \infty)$, $g \in \mathcal{D}(\mathcal{L})$ and $f(t,x)$ be a function defined on $(0, T_0] \times E$ so that for a.e. $t \in (0, T_0]$, $f(t, \cdot) \in \mathcal{D}(\mathcal{L})$and

$$\int_0^{T_0} \|\mathcal{L} f(t,\cdot)\| \, dt < \infty \quad \text{and} \quad \|f(t,\cdot)\| \leq K < \infty \text{ for a.e. } t \in (0, T_0].$$

Then the function

$$\begin{aligned} u(t,x) &:= \mathbb{E}\left[P^0_{S_t^{-1}} g(x)\right] + \mathbb{E}\left[\int_{r=0}^{\infty} \mathbf{1}_{\{S_r < t\}} P_r^0 f(t - S_r, \cdot)(x)\, dr\right] \\ &= \mathbb{E}\left[P^0_{S_t^{-1}} g(x)\right] + \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x) \bar{p}(r, t-s)\, dr\, ds \end{aligned} \tag{2.1}$$

is the unique strong solution of

$$\partial_t^w u(t,x) = \mathcal{L} u(t,x) + f(t,x) \text{ on } (0, T_0] \times E \text{ with } u(0,x) = g(x)$$

in the following sense:

Theorem 1. Existence and uniqueness of Strong solution

Suppose that $\{P_t^0; t \geq 0\}$ is a uniformly bounded and strongly continuous semigroup in some Banach space $(\mathbb{B}, \|\cdot\|)$ over a locally compact separable metric space E and $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is its infinitesimal generator.
Let $T_0 \in (0, \infty)$, $g \in \mathcal{D}(\mathcal{L})$ and $f(t,x)$ be a function defined on $(0, T_0] \times E$ so that for a.e. $t \in (0, T_0]$, $f(t, \cdot) \in \mathcal{D}(\mathcal{L})$and

$$\int_0^{T_0} \|\mathcal{L} f(t,\cdot)\| \, dt < \infty \quad \text{and} \quad \|f(t,\cdot)\| \leq K < \infty \text{ for a.e. } t \in (0, T_0].$$

Then the function

$$\begin{aligned} u(t,x) &:= \mathbb{E}\left[P^0_{S_t^{-1}} g(x)\right] + \mathbb{E}\left[\int_{r=0}^{\infty} \mathbf{1}_{\{S_r < t\}} P_r^0 f(t - S_r, \cdot)(x)\, dr\right] \\ &= \mathbb{E}\left[P^0_{S_t^{-1}} g(x)\right] + \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x) \bar{p}(r, t-s)\, dr\, ds \end{aligned} \tag{2.1}$$

is the unique strong solution of

$$\partial_t^w u(t,x) = \mathcal{L} u(t,x) + f(t,x) \text{ on } (0, T_0] \times E \text{ with } u(0,x) = g(x)$$

in the following sense:

Theorem 1 (continue): $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ on $(0,T_0] \times E$ with $u(0,x) = g(x)$

(i) $u(t,\cdot)$ is well defined as an element in $\mathbb{B}$ for each $t \in (0,T_0]$ such that $\sup_{t\in(0,T_0]} \|u(t,\cdot)\| < \infty$, $t \mapsto u(t,x)$ is continuous in $(\mathbb{B}, \|\cdot\|)$ and

$$\lim_{t\to 0} \|u(t,\cdot) - g(\cdot)\| = 0.$$

(ii) For a.e. $t \in (0,T_0]$, $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}u(t,\cdot)$ exists in the Banach space $\mathbb{B}$ such that $\int_0^{T_0} \|\mathcal{L}u(t,\cdot)\|\, dt < \infty$,

$$\int_0^t w(t-s)\,(u(s,\cdot) - g(\cdot))\, ds = \int_0^t (f(s,\cdot) + \mathcal{L}u(s,\cdot))\, ds \quad \text{in } \mathbb{B}.$$

Theorem 1 (continue): $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ on $(0,T_0] \times E$ with $u(0,x) = g(x)$

(i) $u(t,\cdot)$ is well defined as an element in $\mathbb{B}$ for each $t \in (0,T_0]$ such that $\sup_{t\in(0,T_0]} \|u(t,\cdot)\| < \infty$, $t \mapsto u(t,x)$ is continuous in $(\mathbb{B}, \|\cdot\|)$ and

$$\lim_{t\to 0} \|u(t,\cdot) - g(\cdot)\| = 0.$$

(ii) For a.e. $t \in (0,T_0]$, $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}u(t,\cdot)$ exists in the Banach space $\mathbb{B}$ such that $\int_0^{T_0} \|\mathcal{L}u(t,\cdot)\|\, dt < \infty$,

$$\int_0^t w(t-s)\,(u(s,\cdot)-g(\cdot))\;ds = \int_0^t (f(s,\cdot)+\mathcal{L}u(s,\cdot))\;ds \quad \text{in } \mathbb{B}.$$

Theorem 2. Existence and uniqueness of Weak solution

We will always assume that (E, d) is a locally compact separable metric space with a fully-supported Radon measure μ.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form on $L^2(E; \mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(E; \mu)$.

Suppose that $g \in L^2(E; \mu)$ and that $f(t, x)$ is a function on $(0, T_0] \times E$ so that

$$\|f(t, \cdot)\|_{L^2(E;\mu)} \leq K < \infty \quad \text{for a.e. } t \in (0, T_0].$$

Then $u(t, x)$ defined by (2.1) is the unique weak solution of

$$\partial_t^w u(t, x) = \mathcal{L}u(t, x) + f(t, x) \text{ on } (0, T_0] \times E \text{ with } u(0, x) = g(x)$$

in the following sense:

Theorem 2. Existence and uniqueness of Weak solution

We will always assume that (E, d) is a locally compact separable metric space with a fully-supported Radon measure μ.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form on $L^2(E;\mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(E;\mu)$.

Suppose that $g \in L^2(E;\mu)$ and that $f(t,x)$ is a function on $(0, T_0] \times E$ so that

$$\|f(t,\cdot)\|_{L^2(E;\mu)} \leq K < \infty \quad \text{for a.e. } t \in (0, T_0].$$

Then $u(t,x)$ defined by (2.1) is the unique weak solution of

$$\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x) \text{ on } (0,T_0] \times E \text{ with } u(0,x) = g(x)$$

in the following sense:

Theorem 2. Existence and uniqueness of Weak solution

We will always assume that (E, d) is a locally compact separable metric space with a fully-supported Radon measure μ.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form on $L^2(E; \mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(E; \mu)$.

Suppose that $g \in L^2(E; \mu)$ and that $f(t, x)$ is a function on $(0, T_0] \times E$ so that

$$\|f(t, \cdot)\|_{L^2(E;\mu)} \leq K < \infty \quad \text{for a.e. } t \in (0, T_0].$$

Then $u(t, x)$ defined by (2.1) is the unique weak solution of

$$\partial_t^w u(t, x) = \mathcal{L}u(t, x) + f(t, x) \text{ on } (0, T_0] \times E \text{ with } u(0, x) = g(x)$$

in the following sense:

Theorem 2. Existence and uniqueness of Weak solution

We will always assume that (E, d) is a locally compact separable metric space with a fully-supported Radon measure μ.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form on $L^2(E;\mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(E;\mu)$.

Suppose that $g \in L^2(E;\mu)$ and that $f(t,x)$ is a function on $(0, T_0] \times E$ so that

$$\|f(t,\cdot)\|_{L^2(E;\mu)} \leq K < \infty \text{ for a.e. } t \in (0, T_0].$$

Then $u(t,x)$ defined by (2.1) is the unique weak solution of

$$\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x) \text{ on } (0, T_0] \times E \text{ with } u(0,x) = g(x)$$

in the following sense:

Theorem 2. Existence and uniqueness of Weak solution

We will always assume that (E, d) is a locally compact separable metric space with a fully-supported Radon measure μ.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form on $L^2(E;\mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L^2(E;\mu)$.

Suppose that $g \in L^2(E;\mu)$ and that $f(t,x)$ is a function on $(0, T_0] \times E$ so that

$$\|f(t,\cdot)\|_{L^2(E;\mu)} \leq K < \infty \text{ for a.e. } t \in (0, T_0].$$

Then $u(t,x)$ defined by (2.1) is the unique weak solution of

$$\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x) \text{ on } (0,T_0] \times E \text{ with } u(0,x) = g(x)$$

in the following sense:

Theorem 2 (continue): $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ on $(0,T_0] \times E$ with $u(0,x) = g(x)$

(i) $t \mapsto u(t,x)$ is continuous in $L^2(E;\mu)$,

$$\sup_{t\in(0,T_0]} \|u(t,\cdot)\|_{L^2(E;\mu)} < \infty \quad \text{and} \quad u(t,\cdot) \to g \text{ in } L^2(E;\mu) \text{ as } t \to 0.$$

(ii) For every $t \in (0,T_0]$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\left\langle \int_0^t w(t-s)\,(u(s,\cdot)-g(\cdot))\,ds, \varphi \right\rangle = \int_0^t \langle f(s,\cdot),\varphi\rangle\,ds + \int_0^t \langle u(s,\cdot),\mathcal{L}\varphi\rangle\,ds$$

Theorem 2 (continue): $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ on $(0,T_0] \times E$ with $u(0,x) = g(x)$

(i) $t \mapsto u(t,x)$ is continuous in $L^2(E;\mu)$,

$$\sup_{t\in(0,T_0]} \|u(t,\cdot)\|_{L^2(E;\mu)} < \infty \quad \text{and} \quad u(t,\cdot) \to g \text{ in } L^2(E;\mu) \text{ as } t \to 0.$$

(ii) For every $t \in (0,T_0]$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\left\langle \int_0^t w(t-s)\left(u(s,\cdot) - g(\cdot)\right)\, ds, \varphi \right\rangle = \int_0^t \langle f(s,\cdot), \varphi\rangle\, ds + \int_0^t \langle u(s,\cdot), \mathcal{L}\varphi\rangle\, ds$$

Remarks on proofs

Recall that we have assumed that S_t has a bounded density $\bar{p}(t,\cdot)$ for each $t>0$.

Lemma 1

There is a Borel set $\mathcal{N} \subset (0,\infty)$ having zero Lebesgue measure such that for every $t \in (0,\infty)\setminus\mathcal{N}$,

$$\mathbb{P}(S_s \geq t) = \int_0^s \mathbb{E}\left[w(t-S_r)\mathbf{1}_{\{t\geq S_r\}}\right] dr \quad \text{for every } s>0.$$

so that the inverse subordinator S_t^{-1} has a density function given by

$$\frac{d}{dr}\mathbb{P}(S_t^{-1}\leq r) = \int_0^t w(t-s)\bar{p}(r,s)\, ds, \quad r>0.$$

(Chen, 2017)

Remarks on proofs

Let

$$G^{(S)}(t) := \int_0^\infty \bar{p}(r,t)\,dr$$

be the potential density of the subordinator S,

Lemma 2

Let

$$w * G^{(S)}(t) := \int_0^t w(s)G^{(S)}(t-s)\,ds.$$

Then $w * G^{(S)}(t) \le 1$ *for all* $t > 0$*, and* $w * G^{(S)}(t) = 1$ *for a.e.* $t > 0$*.*

Remarks on proofs

For every $f(t,\cdot)$ with $\int_0^{T_0} \|f(t,\cdot)\|\, dt < \infty$,

$$v(t,x) := \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x) \bar{p}(r, t-s)\, dr\, ds$$

is well defined for a.e. $t \in [0, T_0]$ as an element in $\mathbb{B}$ such that $\int_0^{T_0} \|v(t,x)\|\, dt < \infty$. Moreover, for every $T \in (0, T_0]$,

$$\int_0^T w(T-t) v(t,x)\, dt = \int_0^T \mathbb{E}\left[P^0_{S^{-1}_{T-s}} f(s,\cdot)(x) \right] ds$$

as elements in $\mathbb{B}$.

Outline

We assume that the Banach space $(\mathbb{B}, \|\cdot\|)$ is either $\mathbb{B} = L^p(E;\mu)$ with $p \geq 1$, or $\mathbb{B} = C_\infty(E)$, the space of continuous functions on E that vanish at infinity.

Let $\{P_t^0; t \geq 0\}$ be a uniformly bounded and strongly continuous semigroup in $(\mathbb{B}, \|\cdot\|)$.

Recall that $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with infinite Lévy measure ν such that *the subordinator* S_r *has a bounded density function* $\bar{p}(r, \cdot)$ *for each* $r > 0$.

As a particular case of Theorems 1 and 2, we know that

$$u(t,x) := \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x) \bar{p}(r, t-s)\, dr\, ds \tag{3.1}$$

is the unique solution to the general time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with $u(0,x) = 0$ under suitable conditions on $f(t,x)$.

We assume that the Banach space $(\mathbb{B}, \|\cdot\|)$ is either $\mathbb{B} = L^p(E;\mu)$ with $p \geq 1$, or $\mathbb{B} = C_\infty(E)$, the space of continuous functions on E that vanish at infinity.

Let $\{P_t^0; t \geq 0\}$ be a uniformly bounded and strongly continuous semigroup in $(\mathbb{B}, \|\cdot\|)$.

Recall that $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with infinite Lévy measure ν such that *the subordinator* S_r *has a bounded density function* $\bar{p}(r,\cdot)$ *for each* $r > 0$.

As a particular case of Theorems 1 and 2, we know that

$$u(t,x) := \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x)\bar{p}(r,t-s)\,dr\,ds \tag{3.1}$$

is the unique solution to the general time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with $u(0,x) = 0$ under suitable conditions on $f(t,x)$.

We assume that the Banach space $(\mathbb{B}, \|\cdot\|)$ is either $\mathbb{B} = L^p(E;\mu)$ with $p \geq 1$, or $\mathbb{B} = C_\infty(E)$, the space of continuous functions on E that vanish at infinity.

Let $\{P_t^0; t \geq 0\}$ be a uniformly bounded and strongly continuous semigroup in $(\mathbb{B}, \|\cdot\|)$.

Recall that $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with infinite Lévy measure ν such that *the subordinator S_r has a bounded density function $\bar{p}(r, \cdot)$ for each $r > 0$.*

As a particular case of Theorems 1 and 2, we know that

$$u(t,x) := \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x)\bar{p}(r, t-s)\, dr\, ds \tag{3.1}$$

is the unique solution to the general time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with $u(0,x) = 0$ under suitable conditions on $f(t,x)$.

We assume that the Banach space $(\mathbb{B}, \|\cdot\|)$ is either $\mathbb{B} = L^p(E;\mu)$ with $p \geq 1$, or $\mathbb{B} = C_\infty(E)$, the space of continuous functions on E that vanish at infinity.

Let $\{P_t^0; t \geq 0\}$ be a uniformly bounded and strongly continuous semigroup in $(\mathbb{B}, \|\cdot\|)$.

Recall that $S = \{S_t, \mathbb{P}; t \geq 0\}$ is a driftless subordinator with infinite Lévy measure ν such that *the subordinator* S_r *has a bounded density function* $\bar{p}(r,\cdot)$ *for each* $r > 0$.

As a particular case of Theorems 1 and 2, we know that

$$u(t,x) := \int_{s=0}^{t} \int_{r=0}^{\infty} P_r^0 f(s,\cdot)(x)\bar{p}(r,t-s)\,dr\,ds \tag{3.1}$$

is the unique solution to the general time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with $u(0,x) = 0$ under suitable conditions on $f(t,x)$.

We now assume that $\{P_t^0; t \geq 0\}$ has a density function $p^0(t,x,y)$ with respect to some σ-finite measure μ on E with full support.

Then $u(t,x)$ of (3.1) can be written as

$$u(t,x) = \int_0^t \int_E q(t-s,x,y) f(s,y)\,\mu(dy)\,ds,$$

where

$$q(t,x,y) := \int_0^\infty p^0(r,x,y)\bar{p}(r,t)\,dr.$$

In other words, $q(t,x,y)$ is the fundamental solution for solving the time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with zero initial value. This enables us to establish two-sided estimates for the fundamental solution $q(t,x,y)$ for the time fractional Poisson equation using estimates of $p^0(r,x,y)$ and $\bar{p}(r,t)$ only.

On the other hand, recently in [CKKW0], we have shown that

$$p(t,x,y) := \int_0^\infty p^0(r,x,y)\,d_r\mathbb{P}(S_r \geq t)$$

is the fundamental solution of the homogenous time fractional equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x)$.

We now assume that $\{P^0_t; t \geq 0\}$ has a density function $p^0(t,x,y)$ with respect to some σ-finite measure μ on E with full support.
Then $u(t,x)$ of (3.1) can be written as

$$u(t,x) = \int_0^t \int_E q(t-s,x,y) f(s,y)\, \mu(dy)\, ds,$$

where

$$q(t,x,y) := \int_0^\infty p^0(r,x,y) \bar{p}(r,t)\, dr.$$

In other words, $q(t,x,y)$ is the fundamental solution for solving the time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with zero initial value. This enables us to establish two-sided estimates for the fundamental solution $q(t,x,y)$ for the time fractional Poisson equation using estimates of $p^0(r,x,y)$ and $\bar{p}(r,t)$ only.
On the other hand, recently in [CKKW0], we have shown that

$$p(t,x,y) := \int_0^\infty p^0(r,x,y)\, d_r \mathbb{P}(S_r \geq t)$$

is the fundamental solution of the homogenous time fractional equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x)$.

We now assume that $\{P_t^0; t \geq 0\}$ has a density function $p^0(t,x,y)$ with respect to some σ-finite measure μ on E with full support.
Then $u(t,x)$ of (3.1) can be written as

$$u(t,x) = \int_0^t \int_E q(t-s,x,y) f(s,y)\, \mu(dy)\, ds,$$

where

$$q(t,x,y) := \int_0^\infty p^0(r,x,y)\bar{p}(r,t)\, dr.$$

In other words, $q(t,x,y)$ is the fundamental solution for solving the time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with zero initial value. This enables us to establish two-sided estimates for the fundamental solution $q(t,x,y)$ for the time fractional Poisson equation using estimates of $p^0(r,x,y)$ and $\bar{p}(r,t)$ only.
On the other hand, recently in [CKKW0], we have shown that

$$p(t,x,y) := \int_0^\infty p^0(r,x,y)\, d_r \mathbb{P}(S_r \geq t)$$

is the fundamental solution of the homogenous time fractional equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x)$.

We now assume that $\{P_t^0; t \geq 0\}$ has a density function $p^0(t,x,y)$ with respect to some σ-finite measure μ on E with full support.
Then $u(t,x)$ of (3.1) can be written as

$$u(t,x) = \int_0^t \int_E q(t-s,x,y) f(s,y)\, \mu(dy)\, ds,$$

where

$$q(t,x,y) := \int_0^\infty p^0(r,x,y) \bar{p}(r,t)\, dr.$$

In other words, $q(t,x,y)$ is the fundamental solution for solving the time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with zero initial value. This enables us to establish two-sided estimates for the fundamental solution $q(t,x,y)$ for the time fractional Poisson equation using estimates of $p^0(r,x,y)$ and $\bar{p}(r,t)$ only.

On the other hand, recently in [CKKW0], we have shown that

$$p(t,x,y) := \int_0^\infty p^0(r,x,y)\, d_r \mathbb{P}(S_r \geq t)$$

is the fundamental solution of the homogenous time fractional equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x)$.

We now assume that $\{P_t^0; t \geq 0\}$ has a density function $p^0(t,x,y)$ with respect to some σ-finite measure μ on E with full support.
Then $u(t,x)$ of (3.1) can be written as

$$u(t,x) = \int_0^t \int_E q(t-s,x,y) f(s,y)\,\mu(dy)\,ds,$$

where

$$q(t,x,y) := \int_0^\infty p^0(r,x,y)\bar{p}(r,t)\,dr.$$

In other words, $q(t,x,y)$ is the fundamental solution for solving the time fractional Poisson equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x) + f(t,x)$ with zero initial value. This enables us to establish two-sided estimates for the fundamental solution $q(t,x,y)$ for the time fractional Poisson equation using estimates of $p^0(r,x,y)$ and $\bar{p}(r,t)$ only.
On the other hand, recently in [CKKW0], we have shown that

$$p(t,x,y) := \int_0^\infty p^0(r,x,y)\,d_r\mathbb{P}(S_r \geq t)$$

is the fundamental solution of the homogenous time fractional equation $\partial_t^w u(t,x) = \mathcal{L}u(t,x)$.

The subordinator S is said to be special, if its Laplace exponent ϕ is a special Bernstein function; that is, $\lambda \mapsto \phi^*(\lambda) := \lambda/\phi(\lambda)$ is still a Bernstein function.

In this case, let $S^* = \{S_t^*, \mathbb{P}; t \geq 0\}$ be the subordinator with the Laplace exponent $\phi^*(\lambda)$.

We call S^* the conjugate subordinator to S. Let ν^* be the Lévy measure of S^* and $w^*(x) := \nu^*(x, \infty)$.

See *Bernstein Functions. Theory and Applications* (2nd Edn), 2012 (R. L. Schilling, R. Song and Z. Vondraček) for examples.

The subordinator S is said to be special, if its Laplace exponent ϕ is a special Bernstein function; that is, $\lambda \mapsto \phi^*(\lambda) := \lambda/\phi(\lambda)$ is still a Bernstein function.

In this case, let $S^* = \{S_t^*, \mathbb{P}; t \geq 0\}$ be the subordinator with the Laplace exponent $\phi^*(\lambda)$.

We call S^* the conjugate subordinator to S. Let ν^* be the Lévy measure of S^* and $w^*(x) := \nu^*(x, \infty)$.

See *Bernstein Functions. Theory and Applications* (2nd Edn), 2012 (R. L. Schilling, R. Song and Z. Vondraček) for examples.

The subordinator S is said to be special, if its Laplace exponent ϕ is a special Bernstein function; that is, $\lambda \mapsto \phi^*(\lambda) := \lambda/\phi(\lambda)$ is still a Bernstein function.

In this case, let $S^* = \{S_t^*, \mathbb{P}; t \geq 0\}$ be the subordinator with the Laplace exponent $\phi^*(\lambda)$.

We call S^* the conjugate subordinator to S. Let ν^* be the Lévy measure of S^* and $w^*(x) := \nu^*(x, \infty)$.

See *Bernstein Functions. Theory and Applications* (2nd Edn), 2012 (R. L. Schilling, R. Song and Z. Vondraček) for examples.

The subordinator S is said to be special, if its Laplace exponent ϕ is a special Bernstein function; that is, $\lambda \mapsto \phi^*(\lambda) := \lambda/\phi(\lambda)$ is still a Bernstein function.

In this case, let $S^* = \{S_t^*, \mathbb{P}; t \geq 0\}$ be the subordinator with the Laplace exponent $\phi^*(\lambda)$.

We call S^* the conjugate subordinator to S. Let ν^* be the Lévy measure of S^* and $w^*(x) := \nu^*(x, \infty)$.

See *Bernstein Functions. Theory and Applications* (2nd Edn), 2012 (R. L. Schilling, R. Song and Z. Vondraček) for examples.

The subordinator S is said to be special, if its Laplace exponent ϕ is a special Bernstein function; that is, $\lambda \mapsto \phi^*(\lambda) := \lambda/\phi(\lambda)$ is still a Bernstein function.

In this case, let $S^* = \{S_t^*, \mathbb{P}; t \geq 0\}$ be the subordinator with the Laplace exponent $\phi^*(\lambda)$.

We call S^* the conjugate subordinator to S. Let ν^* be the Lévy measure of S^* and $w^*(x) := \nu^*(x, \infty)$.

See *Bernstein Functions. Theory and Applications* (2nd Edn), 2012 (R. L. Schilling, R. Song and Z. Vondraček) for examples.

Proposition 3.1

Suppose that S is a special subordinator. Then the following two statements hold.

(i) *For any local Lipschitz function f on $[0, \infty)$,*

$$I^{w^*} \circ \partial_t^w f(t) = f(t) - f(0), \quad t \geq 0.$$

In particular, $\partial_t^{w^*} \circ \partial_t^w f(t) = f'(t)$ for a.e. $t > 0$.

(ii) *For any locally integrable function f on $[0, \infty)$,*

$$\partial_t^w \circ I^{w^*} f(t) = f(t), \quad a.e.\ t \geq 0.$$

Proposition 3.1

Suppose that S is a special subordinator. Then the following two statements hold.

(i) *For any local Lipschitz function f on $[0, \infty)$,*

$$I^{w^*} \circ \partial_t^w f(t) = f(t) - f(0), \quad t \geq 0.$$

In particular, $\partial_t^{w^*} \circ \partial_t^w f(t) = f'(t)$ for a.e. $t > 0$.

(ii) *For any locally integrable function f on $[0, \infty)$,*

$$\partial_t^w \circ I^{w^*} f(t) = f(t), \quad a.e. \ t \geq 0.$$

$$q(t,x,y)=\int_0^\infty p^0(r,x,y)\bar{p}(r,t)dr$$
$$p(t,x,y)=\int_0^\infty p^0(r,x,y)\,d_r\mathbb{P}(S_r\geq t)$$

Theorem 3.2

Suppose that S is a special subordinator. Denote by ν^* the Lévy measure of the conjugate subordinator S^* to S and set $w^*(x):=\nu^*(x,\infty)$. Then for μ-a.e. $x,y\in E$,

$$\int_0^t q(s,x,y)\,ds=\int_0^t w^*(t-s)p(s,x,y)\,ds<\infty \quad \text{for } t>0.$$

Hence,

$$q(t,x,y)=\partial_t^{w^*}p(\cdot,x,y)(t) \quad \text{for a.e. } t>0.$$

$$q(t,x,y) = \int_0^\infty p^0(r,x,y)\bar{p}(r,t)dr$$
$$p(t,x,y) = \int_0^\infty p^0(r,x,y)\, d_r\mathbb{P}(S_r \geq t)$$

Theorem 3.2

Suppose that S is a special subordinator. Denote by ν^* the Lévy measure of the conjugate subordinator S^* to S and set $w^*(x) := \nu^*(x,\infty)$.

Then for μ-a.e. $x, y \in E$,

$$\int_0^t q(s,x,y)\, ds = \int_0^t w^*(t-s)p(s,x,y)\, ds < \infty \quad \text{for } t > 0.$$

Hence,

$$q(t,x,y) = \partial_t^{w^*} p(\cdot,x,y)(t) \quad \text{for a.e. } t > 0.$$

$$q(t,x,y) = \int_0^\infty p^0(r,x,y)\bar{p}(r,t)dr$$
$$p(t,x,y) = \int_0^\infty p^0(r,x,y)\, d_r\mathbb{P}(S_r \geq t)$$

Theorem 3.2

Suppose that S is a special subordinator. Denote by ν^* the Lévy measure of the conjugate subordinator S^* to S and set $w^*(x) := \nu^*(x,\infty)$.
Then for μ-a.e. $x,y \in E$,

$$\int_0^t q(s,x,y)\, ds = \int_0^t w^*(t-s)p(s,x,y)\, ds < \infty \quad \textit{for } t > 0.$$

Hence,

$$q(t,x,y) = \partial_t^{w^*} p(\cdot,x,y)(t) \quad \textit{for a.e. } t > 0.$$

Outline

Suppose that the fundamental solution $p^0(t,x,y)$ of $\mathcal{L}$ admits the following two-sided estimates:

$$p^0(t,x,y) \asymp t^{-d/\alpha} F(d(x,y)/t^{1/\alpha}), \tag{4.1}$$

where either

(i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ with $\alpha > 1$.

or

(ii) $F(r) = (1+r)^{-d-\alpha}$, with $\alpha > 0$.

Suppose that the fundamental solution $p^0(t,x,y)$ of $\mathcal{L}$ admits the following two-sided estimates:

$$p^0(t,x,y) \asymp t^{-d/\alpha} F(d(x,y)/t^{1/\alpha}), \tag{4.1}$$

where either

(i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ with $\alpha > 1$.

or

(ii) $F(r) = (1+r)^{-d-\alpha}$, with $\alpha > 0$.

Suppose that the fundamental solution $p^0(t,x,y)$ of $\mathcal{L}$ admits the following two-sided estimates:

$$p^0(t,x,y) \asymp t^{-d/\alpha} F(d(x,y)/t^{1/\alpha}), \tag{4.1}$$

where either

(i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ with $\alpha > 1$.

or

(ii) $F(r) = (1+r)^{-d-\alpha}$, with $\alpha > 0$.

Case (i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ typically corresponds to diffusion case.

When $\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1} I_{d\times d} \le (a_{ij}(x)) \le \lambda I_{d\times d}$ on $\mathbb{R}^d$, it is known due to a result by Aronson (67) that $\mathcal{L}$ admits such an estimate with $\alpha = 2$.

When $\mathcal{L}$ is the Laplacian on a two-dimensional unbounded Sierpinski gasket, it is shown by Barlow and Perkins (88) that the the case (i) hold $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

Case (ii) $F(r) = (1+r)^{-d-\alpha}$ typically corresponds to pure jump processes.

It is shown in Chen and Kumagai (03) that case (ii) hold for symmetric α-stable-like process on Alfhors d-regular space E for $0 < \alpha < 2$.

Case (i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ typically corresponds to diffusion case.

When $\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1} I_{d\times d} \le (a_{ij}(x)) \le \lambda I_{d\times d}$ on $\mathbb{R}^d$, it is known due to a result by Aronson (67) that $\mathcal{L}$ admits such an estimate with $\alpha = 2$.

When $\mathcal{L}$ is the Laplacian on a two-dimensional unbounded Sierpinski gasket, it is shown by Barlow and Perkins (88) that the the case (i) hold $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

Case (ii) $F(r) = (1+r)^{-d-\alpha}$ typically corresponds to pure jump processes.

It is shown in Chen and Kumagai (03) that case (ii) hold for symmetric α-stable-like process on Alfhors d-regular space E for $0 < \alpha < 2$.

Case (i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ typically corresponds to diffusion case.

When $\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1} I_{d\times d} \le (a_{ij}(x)) \le \lambda I_{d\times d}$ on $\mathbb{R}^d$, it is known due to a result by Aronson (67) that $\mathcal{L}$ admits such an estimate with $\alpha = 2$.

When $\mathcal{L}$ is the Laplacian on a two-dimensional unbounded Sierpinski gasket, it is shown by Barlow and Perkins (88) that the the case (i) hold $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

Case (ii) $F(r) = (1+r)^{-d-\alpha}$ typically corresponds to pure jump processes.

It is shown in Chen and Kumagai (03) that case (ii) hold for symmetric α-stable-like process on Alfhors d-regular space E for $0 < \alpha < 2$.

Case (i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ typically corresponds to diffusion case.

When $\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1} I_{d\times d} \le (a_{ij}(x)) \le \lambda I_{d\times d}$ on $\mathbb{R}^d$, it is known due to a result by Aronson (67) that $\mathcal{L}$ admits such an estimate with $\alpha = 2$.

When $\mathcal{L}$ is the Laplacian on a two-dimensional unbounded Sierpinski gasket, it is shown by Barlow and Perkins (88) that the the case (i) hold $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

Case (ii) $F(r) = (1+r)^{-d-\alpha}$ typically corresponds to pure jump processes.

It is shown in Chen and Kumagai (03) that case (ii) hold for symmetric α-stable-like process on Alfhors d-regular space E for $0 < \alpha < 2$.

Case (i) $F(r) = \exp\left(-r^{\alpha/(\alpha-1)}\right)$ typically corresponds to diffusion case.

When $\mathcal{L} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $\lambda^{-1}I_{d\times d} \le (a_{ij}(x)) \le \lambda I_{d\times d}$ on $\mathbb{R}^d$, it is known due to a result by Aronson (67) that $\mathcal{L}$ admits such an estimate with $\alpha = 2$.

When $\mathcal{L}$ is the Laplacian on a two-dimensional unbounded Sierpinski gasket, it is shown by Barlow and Perkins (88) that the the case (i) hold $d = \log 3/\log 2$ and $\alpha = d_w := \log 5/\log 2$.

Case (ii) $F(r) = (1+r)^{-d-\alpha}$ typically corresponds to pure jump processes.

It is shown in Chen and Kumagai (03) that case (ii) hold for symmetric α-stable-like process on Alfhors d-regular space E for $0 < \alpha < 2$.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;\mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Grigor'yan-Kumagai '08

Assume that (E, d) satisfies the chain condition ($\exists C > 0$, $\forall x, y \in M$ $\forall n \in \mathbb{N}$, $\exists \{x_i\}_{i=0}^n \subset E$ s.t. $x_0 = x$, $x_n = y$, and $d(x_i, x_{i+1}) \leq Cd(x,y)/n$) and all balls are relatively compact. Assume further that $(\mathcal{E}, \mathcal{F})$ is regular, conservative and (4.1) holds with some $d, \alpha > 0$ and non-increasing function F.

Then $\alpha \leq d+1$, $\mu\left(B\left(x,r\right)\right) \simeq r^d$ for all $x \in M$ and $r > 0$, and the following dichotomy holds:
either (1) the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local, $\alpha \geq 2$, E is connected, and $F\left(s\right) \asymp \exp\left(-s^{\alpha/(\alpha-1)}\right)$,
or (2) the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of pure jump type and $F\left(s\right) \simeq \left(1+s\right)^{-(d+\alpha)}$.

Note: $\alpha = 2$ and $\Psi(s) = \exp(-s^2)$ is the classical case.

Suppose that $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is the infinitesimal generator of a (symmetric) Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;\mu)$ and $\{P_t^0; t \geq 0\}$ is its associated transition semigroup.

Grigor'yan-Kumagai '08

Assume that (E, d) satisfies the chain condition ($\exists C > 0$, $\forall x, y \in M$ $\forall n \in \mathbb{N}$, $\exists \{x_i\}_{i=0}^n \subset E$ s.t. $x_0 = x$, $x_n = y$, and $d(x_i, x_{i+1}) \leq Cd(x, y)/n$) and all balls are relatively compact. Assume further that $(\mathcal{E}, \mathcal{F})$ is regular, conservative and (4.1) holds with some $d, \alpha > 0$ and non-increasing function F.

Then $\alpha \leq d + 1$, $\mu\left(B\left(x, r\right)\right) \simeq r^d$ for all $x \in M$ and $r > 0$, and the following dichotomy holds:
either (1) the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is local, $\alpha \geq 2$, E is connected, and $F\left(s\right) \asymp \exp\left(-s^{\alpha/(\alpha-1)}\right)$,
or (2) the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is of pure jump type and $F\left(s\right) \simeq \left(1+s\right)^{-(d+\alpha)}$.

Note: $\alpha = 2$ and $\Psi(s) = \exp(-s^2)$ is the classical case.

For $\beta \in (0,1)$, define

$$h(t,r) = \begin{cases} t^{-\beta d/\alpha}, & d < \alpha, \\ t^{-\beta} \log\left(\frac{2t^\beta}{r^\alpha}\right), & d = \alpha, \\ t^{-\beta}/r^{d-\alpha}, & d > \alpha, \end{cases}$$

Theorem 4.1

Suppose that $\{S_t, \mathbb{P}; t \geq 0\}$ *is a* β*-stable subordinator with* $0 < \beta < 1$*.*

(i) *Suppose* $F(s) = (1+s)^{-d-\alpha}$*. Then,*

$$p(t,x,y) \simeq \begin{cases} h(t, d(x,y)) & \text{if } d(x,y) \leq t^{\beta/\alpha}, \\ t^\beta / d(x,y)^{d+\alpha} & \text{if } d(x,y) \geq t^{\beta/\alpha}. \end{cases}$$

(ii) *Suppose* $F(s) = \exp(-s^{\alpha/(\alpha-1)})$ *with* $\alpha \geq 2$*. Then*

$$\begin{aligned} p(t,x,y) &\simeq h(t, d(x,y)) && \text{if } d(x,y) \leq t^{\beta/\alpha}, \\ p(t,x,y) &\asymp t^{-\beta d/\alpha} \exp\left((d(x,y) t^{-\beta/\alpha})^{\alpha/(\alpha-\beta)} \right) && \text{if } d(x,y) \geq t^{\beta/\alpha}. \end{aligned}$$

For $\beta \in (0,1)$, define

$$H(t,r) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-1-\beta} \log\left(\frac{2t^\beta}{r^\alpha}\right), & d = 2\alpha, \\ t^{-1-\beta}/r^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

Theorem 4.2

Suppose that $\{S_t, \mathbb{P}; t \geq 0\}$ *is a* β*-stable subordinator with* $0 < \beta < 1$.

(i) *Suppose* $F(r) = (1+r)^{-d-\alpha}$ *with* $\alpha > 0$. *Then,*

$$q(t,x,y) \simeq \begin{cases} H(t, d(x,y)) & \text{if } d(x,y) \leq t^{\beta/\alpha}, \\ t^{2\beta-1}/d(x,y)^{d+\alpha} & \text{if } d(x,y) \geq t^{\beta/\alpha}. \end{cases}$$

(ii) *Suppose* $F(r) = \exp(-r^{\alpha/(\alpha-1)})$ *with* $\alpha > 1$. *Then,*

$$q(t,x,y) \simeq H(t, d(x,y)) \qquad \text{if } d(x,y) \leq t^{\beta/\alpha},$$

$$q(t,x,y) \asymp t^{\beta-1-\beta d/\alpha} \exp\left(-\left(d(x,y)^\alpha/t^\beta\right)^{1/(\alpha-\beta)}\right) \qquad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

For $\beta \in (0,1)$, define

$$H(t,r) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-1-\beta} \log\left(\frac{2t^\beta}{r^\alpha}\right), & d = 2\alpha, \\ t^{-1-\beta}/r^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

Theorem 4.2

Suppose that $\{S_t, \mathbb{P}; t \geq 0\}$ is a β-stable subordinator with $0 < \beta < 1$.

(i) *Suppose $F(r) = (1+r)^{-d-\alpha}$ with $\alpha > 0$. Then,*

$$q(t,x,y) \simeq \begin{cases} H(t, d(x,y)) & \text{if } d(x,y) \leq t^{\beta/\alpha}, \\ t^{2\beta-1}/d(x,y)^{d+\alpha} & \text{if } d(x,y) \geq t^{\beta/\alpha}. \end{cases}$$

(ii) *Suppose $F(r) = \exp(-r^{\alpha/(\alpha-1)})$ with $\alpha > 1$. Then,*

$$q(t,x,y) \simeq H(t,d(x,y)) \qquad \text{if } d(x,y) \leq t^{\beta/\alpha},$$

$$q(t,x,y) \asymp t^{\beta-1-\beta d/\alpha} \exp\left(-\left(d(x,y)^\alpha/t^\beta\right)^{1/(\alpha-\beta)}\right) \qquad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

For $\beta \in (0,1)$, define

$$H(t,r) = \begin{cases} t^{\beta-1-\beta d/\alpha}, & d < 2\alpha, \\ t^{-1-\beta} \log\left(\frac{2t^\beta}{r^\alpha}\right), & d = 2\alpha, \\ t^{-1-\beta}/r^{d-2\alpha}, & d > 2\alpha, \end{cases}$$

Theorem 4.2

Suppose that $\{S_t, \mathbb{P}; t \geq 0\}$ is a β-stable subordinator with $0 < \beta < 1$.

(i) *Suppose $F(r) = (1+r)^{-d-\alpha}$ with $\alpha > 0$. Then,*

$$q(t,x,y) \simeq \begin{cases} H(t, d(x,y)) & \text{if } d(x,y) \leq t^{\beta/\alpha}, \\ t^{2\beta-1}/d(x,y)^{d+\alpha} & \text{if } d(x,y) \geq t^{\beta/\alpha}. \end{cases}$$

(ii) *Suppose $F(r) = \exp(-r^{\alpha/(\alpha-1)})$ with $\alpha > 1$. Then,*

$$q(t,x,y) \simeq H(t, d(x,y)) \qquad \text{if } d(x,y) \leq t^{\beta/\alpha},$$

$$q(t,x,y) \asymp t^{\beta-1-\beta d/\alpha} \exp\left(-\left(d(x,y)^\alpha/t^\beta\right)^{1/(\alpha-\beta)}\right) \qquad \text{if } d(x,y) \geq t^{\beta/\alpha}.$$

Assumption 4.3

(i) *The Laplace exponent* ϕ *of* S *satisfies that*

$$c_1\kappa^{\beta_1} \le \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \le c_2\kappa^{\beta_2} \qquad \text{for all } \lambda > 0 \text{ and } \kappa \ge 1,$$

where $0 < \beta_1 < \beta_2 < 1$. *Without loss of generality, we assume* $\phi(1) = 1$.
(ii) *The Lévy measure* $\nu(dz)$ *of* S *has a density function* $\nu(z)$ *with respect to the Lebesgue measure such that*

$$t \mapsto t\nu(t) \text{ is non-increasing on } (0, \infty).$$

Assumption 4.3

(i) *The Laplace exponent* ϕ *of* S *satisfies that*

$$c_1\kappa^{\beta_1} \le \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \le c_2\kappa^{\beta_2} \quad \text{for all } \lambda > 0 \text{ and } \kappa \ge 1,$$

where $0 < \beta_1 < \beta_2 < 1$. *Without loss of generality, we assume* $\phi(1) = 1$.
(ii) *The Lévy measure* $\nu(dz)$ *of* S *has a density function* $\nu(z)$ *with respect to the Lebesgue measure such that*

$$t \mapsto t\nu(t) \text{ is non-increasing on } (0, \infty).$$

Remark

$$c_1\kappa^{\beta_1} \le \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \le c_2\kappa^{\beta_2} \quad \text{for all } \lambda > 0 \text{ and } \kappa \ge 1, \tag{4.2}$$

$$t \mapsto t\nu(t) \text{ is non-increasing on } (0,\infty). \tag{4.3}$$

(i) Under (4.2), the Lévy measure ν of S is infinite as $\nu(0,\infty) = \lim_{\lambda\to\infty}\phi(\lambda) = \infty$, excluding compound Poisson processes.

(ii) Under (4.2) and (4.3), that $\nu(t) \simeq t^{-1}\phi(t^{-1})$ for all $t > 0$. (K, Song and Vondracek (13))

(iii) (4.2) and (4.3) together imply that S_r has a density $\bar{p}(r,t)$ so that $\mathbb{P}(S_r \in dt) = \bar{p}(r,t)\,dt$; moreover, $t \mapsto \bar{p}(r,t)$ is smooth for any $r > 0$. (Sato [Theorems 27.13 and 28.4(ii)] (99))

Remark

$$c_1\kappa^{\beta_1} \le \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \le c_2\kappa^{\beta_2} \quad \text{for all } \lambda > 0 \text{ and } \kappa \ge 1, \tag{4.2}$$

$$t \mapsto t\nu(t) \text{ is non-increasing on } (0,\infty). \tag{4.3}$$

(i) Under (4.2), the Lévy measure ν of S is infinite as $\nu(0,\infty) = \lim_{\lambda\to\infty} \phi(\lambda) = \infty$, excluding compound Poisson processes.

(ii) Under (4.2) and (4.3), that $\nu(t) \simeq t^{-1}\phi(t^{-1})$ for all $t > 0$. (K, Song and Vondracek (13))

(iii) (4.2) and (4.3) together imply that S_r has a density $\bar{p}(r,t)$ so that $\mathbb{P}(S_r \in dt) = \bar{p}(r,t)\,dt$; moreover, $t \mapsto \bar{p}(r,t)$ is smooth for any $r > 0$. (Sato [Theorems 27.13 and 28.4(ii)] (99))

Remark

$$c_1\kappa^{\beta_1} \le \frac{\phi(\kappa\lambda)}{\phi(\lambda)} \le c_2\kappa^{\beta_2} \quad \text{for all } \lambda > 0 \text{ and } \kappa \ge 1, \tag{4.2}$$

$$t \mapsto t\nu(t) \text{ is non-increasing on } (0,\infty). \tag{4.3}$$

(i) Under (4.2), the Lévy measure ν of S is infinite as $\nu(0,\infty) = \lim_{\lambda\to\infty} \phi(\lambda) = \infty$, excluding compound Poisson processes.

(ii) Under (4.2) and (4.3), that $\nu(t) \simeq t^{-1}\phi(t^{-1})$ for all $t > 0$. (K, Song and Vondracek (13))

(iii) (4.2) and (4.3) together imply that S_r has a density $\bar{p}(r,t)$ so that $\mathbb{P}(S_r \in dt) = \bar{p}(r,t)\,dt$; moreover, $t \mapsto \bar{p}(r,t)$ is smooth for any $r > 0$. (Sato [Theorems 27.13 and 28.4(ii)] (99))

Examples

(1) Let $0 < \beta_1 < \beta_2 < 1$.

Suppose a positive function $\kappa(\beta, t)$ defined on $[\beta_1, \beta_2] \times (0, \infty)$ satisfies that $c_1^{-1} \le \kappa(\beta, t) \le c_1$ for all $(\beta, t) \in [\beta_1, \beta_2] \times (0, \infty)$, and that $t \mapsto \kappa(\beta, t)$ is non-increasing on $(0, \infty)$ for any fixed $\beta \in [\beta_1, \beta_2]$.

Define

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt, \text{ where } \nu(t) := t^{-1} \int_{\beta_1}^{\beta_2} \kappa(\beta, t) t^{-\beta} \mu_I(d\beta)$$

and μ_I is a finite measure on $[\beta_1, \beta_2]$. Then clearly (4.3) holds.

Furthermore, since

$$\phi(\lambda) \simeq \int_{\beta_1}^{\beta_2} \left(\int_0^\infty (1 - e^{-s}) s^{-1-\beta}\, ds \right) \lambda^\beta \mu_I(d\beta),$$

it is easy to see that (4.2) holds too.

In particular, stable subordinators, which correspond to μ_I being a Dirac measure $\delta_{\{\beta\}}$ for some $\beta \in (0, 1)$ and $\kappa(\beta, t) \equiv c_0$ for all $t > 0$ for some positive constant c_0, and a larger class of mixtures of stable subordinators satisfy both (4.2) and (4.3).

Examples

(1) Let $0 < \beta_1 < \beta_2 < 1$.
Suppose a positive function $\kappa(\beta, t)$ defined on $[\beta_1, \beta_2] \times (0, \infty)$ satisfies that $c_1^{-1} \le \kappa(\beta, t) \le c_1$ for all $(\beta, t) \in [\beta_1, \beta_2] \times (0, \infty)$, and that $t \mapsto \kappa(\beta, t)$ is non-increasing on $(0, \infty)$ for any fixed $\beta \in [\beta_1, \beta_2]$.
Define

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt, \text{ where } \nu(t) := t^{-1} \int_{\beta_1}^{\beta_2} \kappa(\beta, t) t^{-\beta}\, \mu_I(d\beta)$$

and μ_I is a finite measure on $[\beta_1, \beta_2]$. Then clearly (4.3) holds.
Furthermore, since

$$\phi(\lambda) \simeq \int_{\beta_1}^{\beta_2} \left(\int_0^\infty (1 - e^{-s}) s^{-1-\beta}\, ds \right) \lambda^\beta\, \mu_I(d\beta),$$

it is easy to see that (4.2) holds too.
In particular, stable subordinators, which correspond to μ_I being a Dirac measure $\delta_{\{\beta\}}$ for some $\beta \in (0, 1)$ and $\kappa(\beta, t) \equiv c_0$ for all $t > 0$ for some positive constant c_0, and a larger class of mixtures of stable subordinators satisfy both (4.2) and (4.3).

Examples

(1) Let $0 < \beta_1 < \beta_2 < 1$.
Suppose a positive function $\kappa(\beta, t)$ defined on $[\beta_1, \beta_2] \times (0, \infty)$ satisfies that $c_1^{-1} \le \kappa(\beta, t) \le c_1$ for all $(\beta, t) \in [\beta_1, \beta_2] \times (0, \infty)$, and that $t \mapsto \kappa(\beta, t)$ is non-increasing on $(0, \infty)$ for any fixed $\beta \in [\beta_1, \beta_2]$.
Define

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt, \text{ where } \nu(t) := t^{-1} \int_{\beta_1}^{\beta_2} \kappa(\beta, t) t^{-\beta} \mu_I(d\beta)$$

and μ_I is a finite measure on $[\beta_1, \beta_2]$. Then clearly (4.3) holds.
Furthermore, since

$$\phi(\lambda) \simeq \int_{\beta_1}^{\beta_2} \left(\int_0^\infty (1 - e^{-s}) s^{-1-\beta}\, ds \right) \lambda^\beta \mu_I(d\beta),$$

it is easy to see that (4.2) holds too.
In particular, stable subordinators, which correspond to μ_I being a Dirac measure $\delta_{\{\beta\}}$ for some $\beta \in (0, 1)$ and $\kappa(\beta, t) \equiv c_0$ for all $t > 0$ for some positive constant c_0, and a larger class of mixtures of stable subordinators satisfy both (4.2) and (4.3).

Examples

(1) Let $0 < \beta_1 < \beta_2 < 1$.
Suppose a positive function $\kappa(\beta, t)$ defined on $[\beta_1, \beta_2] \times (0, \infty)$ satisfies that $c_1^{-1} \le \kappa(\beta, t) \le c_1$ for all $(\beta, t) \in [\beta_1, \beta_2] \times (0, \infty)$, and that $t \mapsto \kappa(\beta, t)$ is non-increasing on $(0, \infty)$ for any fixed $\beta \in [\beta_1, \beta_2]$.
Define

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt, \text{ where } \nu(t) := t^{-1} \int_{\beta_1}^{\beta_2} \kappa(\beta, t) t^{-\beta}\, \mu_I(d\beta)$$

and μ_I is a finite measure on $[\beta_1, \beta_2]$. Then clearly (4.3) holds.
Furthermore, since

$$\phi(\lambda) \simeq \int_{\beta_1}^{\beta_2} \left(\int_0^\infty (1 - e^{-s}) s^{-1-\beta}\, ds \right) \lambda^\beta\, \mu_I(d\beta),$$

it is easy to see that (4.2) holds too.
In particular, stable subordinators, which correspond to μ_I being a Dirac measure $\delta_{\{\beta\}}$ for some $\beta \in (0, 1)$ and $\kappa(\beta, t) \equiv c_0$ for all $t > 0$ for some positive constant c_0, and a larger class of mixtures of stable subordinators satisfy both (4.2) and (4.3).

Examples

(1) Let $0 < \beta_1 < \beta_2 < 1$.
Suppose a positive function $\kappa(\beta, t)$ defined on $[\beta_1, \beta_2] \times (0, \infty)$ satisfies that $c_1^{-1} \leq \kappa(\beta, t) \leq c_1$ for all $(\beta, t) \in [\beta_1, \beta_2] \times (0, \infty)$, and that $t \mapsto \kappa(\beta, t)$ is non-increasing on $(0, \infty)$ for any fixed $\beta \in [\beta_1, \beta_2]$.
Define

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt, \text{ where } \nu(t) := t^{-1} \int_{\beta_1}^{\beta_2} \kappa(\beta, t) t^{-\beta} \mu_I(d\beta)$$

and μ_I is a finite measure on $[\beta_1, \beta_2]$. Then clearly (4.3) holds.
Furthermore, since

$$\phi(\lambda) \simeq \int_{\beta_1}^{\beta_2} \left(\int_0^\infty (1 - e^{-s}) s^{-1-\beta}\, ds \right) \lambda^\beta \mu_I(d\beta),$$

it is easy to see that (4.2) holds too.
In particular, stable subordinators, which correspond to μ_I being a Dirac measure $\delta_{\{\beta\}}$ for some $\beta \in (0, 1)$ and $\kappa(\beta, t) \equiv c_0$ for all $t > 0$ for some positive constant c_0, and a larger class of mixtures of stable subordinators satisfy both (4.2) and (4.3).

Examples

(1) Let $0 < \beta_1 < \beta_2 < 1$.
Suppose a positive function $\kappa(\beta, t)$ defined on $[\beta_1, \beta_2] \times (0, \infty)$ satisfies that $c_1^{-1} \le \kappa(\beta, t) \le c_1$ for all $(\beta, t) \in [\beta_1, \beta_2] \times (0, \infty)$, and that $t \mapsto \kappa(\beta, t)$ is non-increasing on $(0, \infty)$ for any fixed $\beta \in [\beta_1, \beta_2]$.
Define

$$\phi(\lambda) := \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt, \text{ where } \nu(t) := t^{-1} \int_{\beta_1}^{\beta_2} \kappa(\beta, t) t^{-\beta}\, \mu_I(d\beta)$$

and μ_I is a finite measure on $[\beta_1, \beta_2]$. Then clearly (4.3) holds.
Furthermore, since

$$\phi(\lambda) \simeq \int_{\beta_1}^{\beta_2} \left(\int_0^\infty (1 - e^{-s}) s^{-1-\beta}\, ds \right) \lambda^\beta\, \mu_I(d\beta),$$

it is easy to see that (4.2) holds too.
In particular, stable subordinators, which correspond to μ_I being a Dirac measure $\delta_{\{\beta\}}$ for some $\beta \in (0, 1)$ and $\kappa(\beta, t) \equiv c_0$ for all $t > 0$ for some positive constant c_0, and a larger class of mixtures of stable subordinators satisfy both (4.2) and (4.3).

Examples

(2) A function $f : (0, \infty) \to [0, \infty)$ is said to be completely monotone, if f is of class C^∞ and $(-1)^n f^{(n)} \geq 0$ on $(0, \infty)$ for every integer $n \geq 0$.

A Bernstein function is said to be a complete Bernstein function, if its Lévy measure has a completely monotone density with respect to the Lebesgue measure.

A sufficient condition on ϕ which implies (4.3) is that ϕ is a Thorin-Bernstein function; that is, both $\phi(\lambda)$ and $\lambda\phi'(\lambda)$ are complete Bernstein functions.

In this case, both $t \mapsto \nu(t)$ and $t \mapsto t\nu(t)$ are completely monotone.

Examples

(2) A function $f : (0, \infty) \to [0, \infty)$ is said to be completely monotone, if f is of class C^∞ and $(-1)^n f^{(n)} \geq 0$ on $(0, \infty)$ for every integer $n \geq 0$.

A Bernstein function is said to be a complete Bernstein function, if its Lévy measure has a completely monotone density with respect to the Lebesgue measure.

A sufficient condition on ϕ which implies (4.3) is that ϕ is a Thorin-Bernstein function; that is, both $\phi(\lambda)$ and $\lambda\phi'(\lambda)$ are complete Bernstein functions.

In this case, both $t \mapsto \nu(t)$ and $t \mapsto t\nu(t)$ are completely monotone.

Examples

(2) A function $f : (0, \infty) \to [0, \infty)$ is said to be completely monotone, if f is of class C^∞ and $(-1)^n f^{(n)} \geq 0$ on $(0, \infty)$ for every integer $n \geq 0$.

A Bernstein function is said to be a complete Bernstein function, if its Lévy measure has a completely monotone density with respect to the Lebesgue measure.

A sufficient condition on ϕ which implies (4.3) is that ϕ is a Thorin-Bernstein function; that is, both $\phi(\lambda)$ and $\lambda\phi'(\lambda)$ are complete Bernstein functions.

In this case, both $t \mapsto \nu(t)$ and $t \mapsto t\nu(t)$ are completely monotone.

Examples

(2) A function $f : (0, \infty) \to [0, \infty)$ is said to be completely monotone, if f is of class C^∞ and $(-1)^n f^{(n)} \geq 0$ on $(0, \infty)$ for every integer $n \geq 0$.

A Bernstein function is said to be a complete Bernstein function, if its Lévy measure has a completely monotone density with respect to the Lebesgue measure.

A sufficient condition on ϕ which implies (4.3) is that ϕ is a Thorin-Bernstein function; that is, both $\phi(\lambda)$ and $\lambda\phi'(\lambda)$ are complete Bernstein functions.

In this case, both $t \mapsto \nu(t)$ and $t \mapsto t\nu(t)$ are completely monotone.

Proposition 4.4

(i) *For each* $L > 0$*, there exist constants* $c_{1,L}, c_{2,L} > 0$ *such that for all* $r\phi(t^{-1}) \le L$*,*

$$c_{1,L} r\phi(t^{-1}) \le \mathbb{P}(S_r \ge t) \le c_{2,L} r\phi(t^{-1}).$$

(ii) *There is a constant* $c_1 > 0$ *such that for all* $r, t > 0$*,*

$$\mathbb{P}(S_r \le t) \le \exp(-c_1 r\phi \circ [(\phi')^{-1}](t/r)) \le \exp(-c_1 t(\phi')^{-1}(t/r)).$$

Moreover, there is a constant $c_0 > 0$ *such that for each* $L > 0$*, there exists a constant* $c_{c_0,L} > 0$ *so that for* $r\phi(t^{-1}) > L$

$$\mathbb{P}(S_r \le t) \ge c_{c_0,L} \exp\left(-c_0 r\phi \circ [(\phi')^{-1}](t/r)\right) \ge c_{c_0,L} \exp(-c_0 C_* t(\phi')^{-1}(t/r)),$$

where $C_* > 0$ *is an absolute constant only depending on* ϕ*.*

Lemma 3

There exists a constant $a_r > 0$ such that for all $r > 0$,
$t \mapsto \bar{p}(r,t)$ is strictly increasing on $[0, a_r)$ and
$t \mapsto \bar{p}(r,t)$ is strictly decreasing on (a_r, ∞).
Moreover, there exists a constant $c_1 \geq 1$ such that

$$c_1^{-1}/\phi^{-1}(1/r) \leq a_r \leq c_1/\phi^{-1}(1/r) \quad \text{for all } r > 0.$$

Theorem 4.5

For each $L > 0$, there exist constants $c_i := c_{i,L} \geq 1$ $(i = 1, 2, 3)$ such that

$$\frac{1}{c_1 t} r\phi(t^{-1}) \leq \bar{p}(r,t) \leq \frac{c_1}{t} r\phi(t^{-1}) \quad \forall r, t > 0 \text{ with } r\phi(t^{-1}) \leq L$$

and

$$\frac{1}{c_2 t} e^{-c_3 t(\phi')^{-1}(t/r)} \leq \bar{p}(r,t) \leq \frac{c_2}{t} e^{-c_3^{-1} t(\phi')^{-1}(t/r)} \quad \forall r, t > 0 \text{ with } r\phi(t^{-1}) \geq L.$$

Suppose that $0 < \alpha_1 \leq \alpha_2 < \infty$. We say that a non-decreasing function $\Psi : (0, \infty) \to (0, \infty)$ satisfies the *weak scaling property with* (α_1, α_2) if there exist constants c_1 and $c_2 > 0$ such that

$$c_1 (R/r)^{\alpha_1} \leq \Psi(R)/\Psi(r) \leq c_2 (R/r)^{\alpha_2} \quad \text{for all } 0 < r \leq R < \infty. \tag{4.4}$$

We say that a family of non-decreasing functions $\{\Psi_x\}_{x\in\Lambda}$ satisfies the *weak scaling property uniformly with* (α_1, α_2) if each Ψ_x satisfies the weak scaling property with constants $c_1, c_2 > 0$ and $0 < \alpha_1 \leq \alpha_2 < \infty$ independent of the choice of $x \in \Lambda$.

Suppose that $0 < \alpha_1 \le \alpha_2 < \infty$. We say that a non-decreasing function $\Psi : (0, \infty) \to (0, \infty)$ satisfies the *weak scaling property with* (α_1, α_2) if there exist constants c_1 and $c_2 > 0$ such that

$$c_1 (R/r)^{\alpha_1} \le \Psi(R)/\Psi(r) \le c_2 (R/r)^{\alpha_2} \quad \text{for all } 0 < r \le R < \infty. \tag{4.4}$$

We say that a family of non-decreasing functions $\{\Psi_x\}_{x \in \Lambda}$ satisfies the *weak scaling property uniformly with* (α_1, α_2) if each Ψ_x satisfies the weak scaling property with constants $c_1, c_2 > 0$ and $0 < \alpha_1 \le \alpha_2 < \infty$ independent of the choice of $x \in \Lambda$.

For $x \in E$ and $r \geq 0$, define

$$V(x,r) = \mu(B(x,r)).$$

We further assume that for each $x \in E$, $V(x,\cdot)$ satisfies the weak scaling property uniformly with (d_1,d_2) for some $d_2 \geq d_1 > 0$; that is, for any $0 < r \leq R$ and $x \in E$,

$$c_1 \left(\frac{R}{r}\right)^{d_1} \leq \frac{V(x,R)}{V(x,r)} \leq c_2 \left(\frac{R}{r}\right)^{d_2}. \tag{4.5}$$

Note that (4.5) is equivalent to the so-called volume doubling and reverse volume doubling conditions.

For $x \in E$ and $r \geq 0$, define

$$V(x, r) = \mu(B(x, r)).$$

We further assume that for each $x \in E$, $V(x, \cdot)$ satisfies the weak scaling property uniformly with (d_1, d_2) for some $d_2 \geq d_1 > 0$; that is, for any $0 < r \leq R$ and $x \in E$,

$$c_1 \left(\frac{R}{r}\right)^{d_1} \leq \frac{V(x, R)}{V(x, r)} \leq c_2 \left(\frac{R}{r}\right)^{d_2}. \tag{4.5}$$

Note that (4.5) is equivalent to the so-called volume doubling and reverse volume doubling conditions.

For $x \in E$ and $r \geq 0$, define

$$V(x,r) = \mu(B(x,r)).$$

We further assume that for each $x \in E$, $V(x,\cdot)$ satisfies the weak scaling property uniformly with (d_1, d_2) for some $d_2 \geq d_1 > 0$; that is, for any $0 < r \leq R$ and $x \in E$,

$$c_1 \left(\frac{R}{r}\right)^{d_1} \leq \frac{V(x,R)}{V(x,r)} \leq c_2 \left(\frac{R}{r}\right)^{d_2}. \tag{4.5}$$

Note that (4.5) is equivalent to the so-called volume doubling and reverse volume doubling conditions.

Pure jump case

We assume that

$$p^0(t,x,y) \simeq \frac{1}{V(x,\Phi^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\Phi(d(x,y))}, \quad t>0, x,y\in E.$$

Here $\Phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\Phi(0)=0$ that satisfies the weak scaling property with (α_1,α_2). (See Chen, Kumagai & Wang ,16+.)

Theorem 4.6

(i) *If* $\Phi(d(x,y))\phi(t^{-1}) \le 1$, *then*

$$p(t,x,y) \simeq \phi(t^{-1}) \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{1}{V(x,\Phi^{-1}(r))}\, dr.$$

(ii) *If* $\Phi(d(x,y))\phi(t^{-1}) \ge 1$, *then* $p(t,x,y) \simeq \dfrac{1}{\phi(t^{-1})V(x,d(x,y))\,\Phi(d(x,y))}$.

Pure jump case

Theorem 4.7

(i) *If* $\Phi(d(x,y))\phi(t^{-1}) \le 1$, *then*

$$q(t,x,y) \simeq \frac{\phi(t^{-1})}{t} \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{r}{V(x,\Phi^{-1}(r))}\, dr. \qquad (4.6)$$

(ii) *If* $\Phi(d(x,y))\phi(t^{-1}) \ge 1$, *then*

$$q(t,x,y) \simeq \frac{1}{t\phi(t^{-1})^2 V(x,d(x,y))\,\Phi(d(x,y))}.$$

Pure jump case

Theorem 4.7

(i) *If* $\Phi(d(x,y))\phi(t^{-1}) \le 1$, *then*

$$q(t,x,y) \simeq \frac{\phi(t^{-1})}{t} \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{r}{V(x,\Phi^{-1}(r))}\, dr. \tag{4.6}$$

(ii) *If* $\Phi(d(x,y))\phi(t^{-1}) \ge 1$, *then*

$$q(t,x,y) \simeq \frac{1}{t\phi(t^{-1})^2 V(x,d(x,y))\,\Phi(d(x,y))}.$$

Pure jump case

Theorem 4.7

(i) *If* $\Phi(d(x,y))\phi(t^{-1}) \leq 1$, *then*

$$q(t,x,y) \simeq \frac{\phi(t^{-1})}{t} \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{r}{V(x,\Phi^{-1}(r))}\, dr. \tag{4.6}$$

(ii) *If* $\Phi(d(x,y))\phi(t^{-1}) \geq 1$, *then*

$$q(t,x,y) \simeq \frac{1}{t\phi(t^{-1})^2 V(x,d(x,y))\,\Phi(d(x,y))}.$$

Diffusion case

We assume that the heat kernel $p^0(t,x,y)$ of the diffusion X with respect to μ exists and enjoys the following two-sided estimates

$$p^0(t,x,y) \asymp \frac{1}{V(x,\Phi^{-1}(t))} \exp\left(-m(t,d(x,y))\right), \quad t>0, x,y \in E.$$

Here, $\Phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\Phi(0)=0$, and satisfies the weak scaling property with (α_1, α_2) such that $\alpha_2 \geq \alpha_1 > 1$

and the function $m(t,r)$ is strictly positive for all $t,r>0$, non-increasing on $t \in (0,\infty)$ for fixed $r>0$, and determined by

$$\frac{t}{m(t,r)} \simeq \Phi\left(\frac{r}{m(t,r)}\right), \quad t,r>0.$$

Diffusion case

We assume that the heat kernel $p^0(t,x,y)$ of the diffusion X with respect to μ exists and enjoys the following two-sided estimates

$$p^0(t,x,y) \asymp \frac{1}{V(x,\Phi^{-1}(t))} \exp\left(-m(t,d(x,y))\right), \quad t>0, x,y \in E.$$

Here, $\Phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\Phi(0)=0$, and satisfies the weak scaling property with (α_1, α_2) such that $\alpha_2 \geq \alpha_1 > 1$

and the function $m(t,r)$ is strictly positive for all $t,r>0$, non-increasing on $t \in (0,\infty)$ for fixed $r>0$, and determined by

$$\frac{t}{m(t,r)} \simeq \Phi\left(\frac{r}{m(t,r)}\right), \quad t,r>0.$$

Diffusion case

We assume that the heat kernel $p^0(t,x,y)$ of the diffusion X with respect to μ exists and enjoys the following two-sided estimates

$$p^0(t,x,y) \asymp \frac{1}{V(x,\Phi^{-1}(t))} \exp\left(-m(t,d(x,y))\right), \quad t>0, x,y \in E.$$

Here, $\Phi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function with $\Phi(0)=0$, and satisfies the weak scaling property with (α_1,α_2) such that $\alpha_2 \geq \alpha_1 > 1$

and the function $m(t,r)$ is strictly positive for all $t,r>0$, non-increasing on $t \in (0,\infty)$ for fixed $r>0$, and determined by

$$\frac{t}{m(t,r)} \simeq \Phi\left(\frac{r}{m(t,r)}\right), \quad t,r>0.$$

Diffusion case

Theorem 4.8

(i) *If* $\Phi(d(x,y))\phi(t^{-1}) \le 1$, *then*

$$p(t,x,y) \simeq \phi(t^{-1}) \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{1}{V(x,\Phi^{-1}(r))}\,dr.$$

(ii) *If* $\Phi(d(x,y))\phi(t^{-1}) \ge 1$, *then there exist constants* $c_i > 0$ $(i = 1, \dots, 4)$ *such that*

$$\frac{c_1}{V(x,\Phi^{-1}(1/\phi(t^{-1})))} \exp(-c_2 n(t,d(x,y))) \le p(t,x,y) \le \frac{c_3}{V(x,\Phi^{-1}(1/\phi(t^{-1})))} \exp(-c_4 n(t,d(x,y))),$$

where $n(\cdot,r)$ *is a non-increasing function on* $(0,\infty)$ *determined by*

$$\frac{1}{\phi(n(t,r)/t)} \simeq \Phi\left(\frac{r}{n(t,r)}\right), \quad t,r>0.$$

Diffusion case

Theorem 4.9

(i) *If* $\Phi(d(x,y))\phi(t^{-1}) \le 1$, *then*

$$q(t,x,y) \simeq \frac{\phi(t^{-1})}{t} \int_{\Phi(d(x,y))}^{2/\phi(t^{-1})} \frac{r}{V(x,\Phi^{-1}(r))}\, dr.$$

(ii) *If* $\Phi(d(x,y))\phi(t^{-1}) \ge 1$, *then there exist constants* $c_i > 0$ $(i = 1, \dots, 4)$ *such that*

$$\frac{c_1}{tV(x,\Phi^{-1}(1/\phi(t^{-1})))} \frac{n(t,d(x,y))}{\phi(n(t,d(x,y))/t)} \exp(-c_2 n(t,d(x,y)))$$

$$\le q(t,x,y) \le \frac{c_3}{tV(x,\Phi^{-1}(1/\phi(t^{-1})))} \frac{n(t,d(x,y))}{\phi(n(t,d(x,y))/t)} \exp(-c_4 n(t,d(x,y))),$$

where $n(\cdot,r)$ *is a non-increasing function on* $(0,\infty)$ *determined by*

$$\frac{1}{\phi(n(t,r)/t)} \simeq \Phi\left(\frac{r}{n(t,r)}\right), \quad t,r>0.$$

Thank you.

감사합니다.