Intermittency for some fractional stochastic heat equations on bounded domains

Eulalia Nualart
(Universitat Pompeu Fabra, Barcelona)

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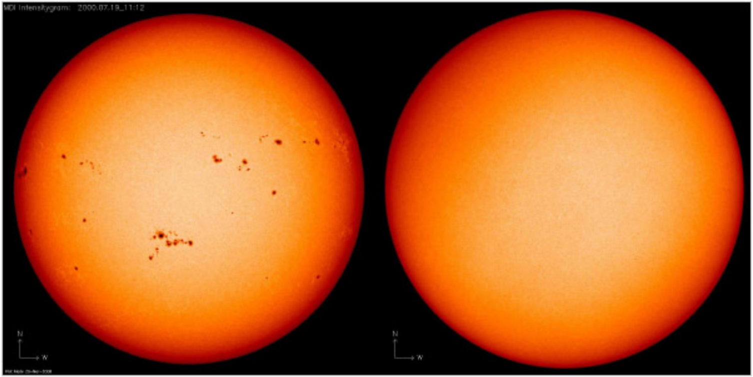
Intermittency of random fields

- Intermittency is a physical phenomena that a random field possesses when it shows widely separated high peaks.
- The most well-known field exhibiting this property is the magnetic field energy in a star.
- In our sun, this exhibits itself as sun spots where most of the magnetic field energy is concentrated, thereby lowering the temperature and causing the darkening which appears as a spot.
- Sunspots may last anywhere from a few days to a few months, but all do eventually decay and disappear.

Is the Sun Missing Its Spots?

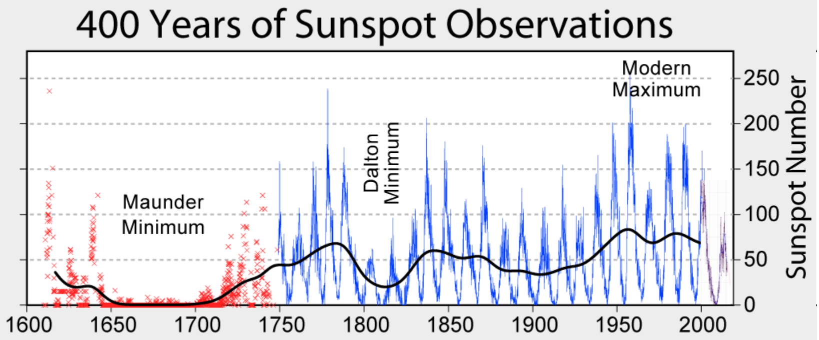


NASA

SUN GAZING These photos show sunspots near solar maximum on July 19, 2000, and near solar minimum on March 18, 2009. Some global warming skeptics speculate that the Sun may be on the verge of an extended slumber.

By KENNETH CHANG
Published: July 20, 2009

Number of sunspots



A 1-D heat equation

$$\begin{aligned} \text{PDE :} \quad & \partial_t u = \frac{1}{2}\partial^2_{xx} u, \quad t > 0, \quad 0 < x < 1, \\ \text{IC :} \quad & u_0(x) = \sin(\pi x), \\ \text{DBC :} \quad & u_t(0) = u_t(1) = 0. \end{aligned}$$

- $u_t(x) =$temperature through a very thin slice of a rod of lenght 1 lying on the x-axis from 0 to 1.
- Since the end of the rods are kept at 0°, we expect that $u \to 0$ as $t \to \infty$.
- The unique solution is :

$$u_t(x) = \sin(\pi x) \exp\left(-\pi^2 t/2\right),$$

so indeed $u \to 0$ as $t \to \infty$.

Heat equation (Khoshnevisan-Kim'15)

$\partial_t u = \frac{1}{2}\partial_x^2 u$ on $[0,1]$ with Dirichlet BC, $u_0(x) = \sin(\pi x)$

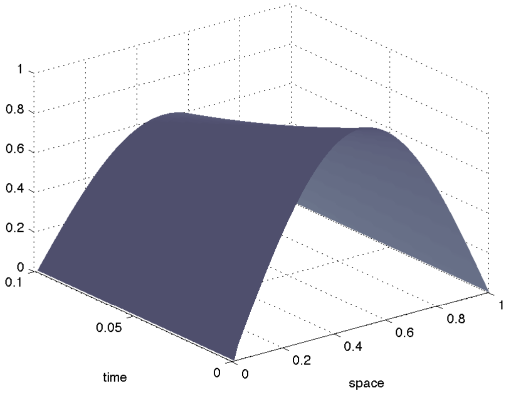


Figure: $\lambda = 0$; $u_t(x) = \sin(\pi x)\exp(-\pi^2 t/2)$

General 1-D heat equation

$$\begin{aligned} \text{PDE}: \quad & \partial_t u = \partial^2_{xx} u, \quad t > 0, \quad 0 < x < 1, \\ \text{IC}: \quad & u_0(x) = f(x) \\ \text{DBC}: \quad & u_t(0) = u_t(1) = 0. \end{aligned}$$

- General solution :
$$u_t(x) = \sum_{n=1}^{\infty} b_n \Phi_n(x) \exp\left(-\mu_n t\right),$$
where $\mu_n = n^2\pi^2$, $\Phi_n(x) = \sqrt{2}\sin(n\pi x)$, and $b_n = \int_0^1 \Phi_n(x) f(x) dx$.
- μ_n and Φ_n are the eigenvalues and eigenfunctions of the Sturm-Liouville problem :
$$\begin{aligned} & X''(x) = -\mu X(x), \quad 0 < x < 1 \\ & X(0) = X(1) = 0. \end{aligned}$$
- So again $u \to 0$ as $t \to \infty$.

A non-linear 1-D heat equation

The cable equation :

$$\begin{aligned} &\text{PDE}: \quad \partial_t u = \partial^2_{xx} u + \lambda u, \quad \lambda > 0, \quad t > 0, \quad 0 < x < 1, \\ &\text{IC}: \quad u_0(x) = \sin(\pi x) \\ &\text{DBC}: \quad u_t(0) = u_t(1) = 0. \end{aligned}$$

- When $\lambda = 1$, $u_t(x)$ represents the electrical potential through an electrical cable. Used for e.g. in the study of neurons.
- General solution :
$$u_t(x) = \sin(\pi x) \exp\left((-\pi^2 + \lambda)t\right),$$
- When $\lambda > \pi^2$, $u \to +\infty$ as $t \to \infty$.
- When $\lambda < \pi^2$, $u \to 0$ as $t \to \infty$.
- When we add a potential, if the potential is large enough it will beat the boundary conditions !

General non-linear heat equation on a bounded domain

$$
\begin{aligned}
\text{PDE}: &\quad \partial_t u = \Delta u + \lambda u, \quad \lambda > 0, \quad t > 0, \quad x \in \mathcal{O}, \\
\text{IC}: &\quad u_0(x) = f(x), \quad f \in L^2(\mathcal{O}) \\
\text{DBC}: &\quad u_t(x) = 0, \quad x \in \partial\mathcal{O}.
\end{aligned}
$$

- Let $\{e_k\}_{k\geq 0}$ be a complete orthonormal system of $L^2(\mathcal{O})$ such that

$$
\left|\begin{array}{l}
\Delta e_k(x) = -\mu_k e_k(x) \quad x \in \mathcal{O} \\
e_k(x) = 0 \quad x \in \partial\mathcal{O},
\end{array}\right.
$$

  where $\{\mu_k\}_{k\geq 0}$ is an increasing sequence of positive numbers.
- In particular, $f = \sum_{k=0}^{\infty} \langle f, e_k \rangle e_k$ and the solution is given by

$$
u_t(x) = \sum_{k=0}^{\infty} \langle f, e_k \rangle e_k(x) \exp\left((-\mu_k + \lambda)t\right),
$$

- Kwiecinsaka'99 : If k_0 is the smallest integer such that $\langle f, e_{k_0} \rangle \neq 0$, then

$$
\limsup_{t\to\infty} \frac{1}{t} \log \|u_t\|_{L^2(\mathcal{O})} = \lambda - \mu_{k_0}.
$$

Gaussian perturbation in time

- Same equation as before but

$$\partial_t u_t(x) = \Delta u_t(x) + \lambda u_t(x) dW_t,$$

- W_t is a real-valued Wiener process. Then the solution is

$$u_t(x) = \exp(\lambda W_t) \sum_{k=0}^{\infty} \langle f, e_k \rangle e_k(x) \exp\left((-\mu_k - \frac{\lambda^2}{2})t\right),$$

- Kwiecinsaka'99 :

$$\limsup_{t\to\infty} \frac{1}{t} \log \|u_t\|_{L^2(\mathcal{O})} = -\frac{\lambda^2}{2} - \mu_{k_0} \quad \text{a.s.}$$

- Using similar computations, one can show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathcal{E}_t(\lambda) = \frac{\lambda^2}{2} - \mu_{k_0},$$

where $\mathcal{E}_t(\lambda) := \sqrt{\mathrm{E}\|u_t\|^2_{L^2(\mathcal{O})}}$.

A non-linear 1-D stochastic heat equation

The Parabolic Anderson Model :

$$\begin{aligned}
&\text{SPDE}: \quad \partial_t u = \partial^2_{xx} u + \lambda u \dot{W}, \quad \lambda > 0, \quad t > 0, \quad 0 < x < 1,\\
&\text{IC}: \quad u_0(x) = f(x),\\
&\text{DBC}: \quad u_t(0) = u_t(1) = 0.
\end{aligned}$$

- When $\lambda = 1$, $u_t(x)$ describes a random mass transport through a magnetic random field of sinks and sources.

- $\dot{W}(t,x)$ space-time white noise. $W(t,x)$ is a centered Gaussian field with covariance
$$\mathrm{E}\left(W(t,x)W(s,y)\right) = (t \wedge s)(x \wedge y).$$

- This process is not differentiable in t or x, so we will rewritte the SPDE in an integral form (Walsh'86) and understand $\dot{W}(t,x)$ as $W(dt,dx)$, where W is a Gaussian random measure.

The mild formulation of Walsh

- The mild solution of the SPDE is the solution to the integral equation

$$u_t(x) = \int_0^1 p_t(x,y) f(y) dy + \lambda \int_0^t \int_0^1 p_{t-s}(x,y) u_s(y) W(ds,dy),$$

where $p_t(x,y)$ is the Dirichlet heat kernel, that is, the solution to the 1-D heat equation with initial condition $f(x) = \delta_y(x)$.

- The stochastic integral was defined by Walsh'86 by extending the L^2-theory of Itô'44 for Brownian motion. It satisfies the following isometry property :

$$\mathrm{E}\left(\left\{\int_0^t \int_0^1 p_{t-s}(x,y) u_s(y) W(ds,dy)\right\}^2\right) = \int_0^t \int_0^1 p_{t-s}^2(x,y) \mathrm{E}\left(u_s^2(y)\right) dsdy$$

Justification of the mild formulation

- Let $\varphi_t(x)$ be a test function with $\varphi_t(0) = \varphi_t(1) = 0$.
- Multiplying the SPDE by $\varphi_t(x)$ and integrating by parts yields

$$\int_0^1 [u_t(x)\varphi_t(x) - u_0(x)\varphi_0(x)]\, dx = \int_0^t \int_0^1 u_s(x)(\partial^2_{xx}\varphi + \partial_t\varphi)_s(x) dxds + \lambda \int_0^t \int_0^1 u_s(x)\varphi_s(x) W(ds, dx).$$

- Set $\varphi_s(y) = \int_0^1 p_{t-s}(x, y)\phi(x)dx$, where ϕ is a test function with $\phi(0) = \phi(1) = 0$. Then, $\varphi_t(y) = \phi(y)$. Set $p_t(\phi, y) = \int_0^1 p_t(x, y)\phi(x)dx$. Integrating by parts,

$$p_t(\phi, y) - p_0(\phi, y) = \int_0^t p_s(\phi'', y)ds.$$

  Thus, $\partial^2_{xx}\varphi + \partial_s\varphi = 0$. Therefore, we obtain

$$\int_0^1 u_t(x)\phi(x)dx = \int_0^1 u_0(y)p_t(\phi, y)dy + \lambda \int_0^t \int_0^1 p_{t-s}(\phi, y)u_s(y)W(ds, dy).$$

- Choosing ϕ as an approximation of the delta function yields the mild formulation.

Existence and uniqueness

$$\begin{aligned} &\text{SPDE}: \quad \partial_t u = \partial^2_{xx} u + \lambda u \dot{W}, \quad \lambda > 0, \quad t > 0, \quad 0 < x < 1 \\ &\text{IC}: \quad u_0(x) = f(x), \\ &\text{DBC}: \quad u_t(0) = u_t(1) = 0. \end{aligned}$$

Theorem (Walsh'86)

Assume f measurable and bounded. There exists a unique predictable jointly measurable and adapted process $(u_t(x), t \geq 0, x \in [0,1])$ *satisfying the integral equation*

$$u_t(x) = \int_0^1 p_t(x,y) f(y) dy + \lambda \int_0^t \int_0^1 p_{t-s}(x,y) u_s(y) W(ds, dy).$$

Moreover, for all $p \geq 2$ *and* $T > 0$*,*

$$\sup_{x \in [0,1]} \sup_{t \in [0,T]} \mathrm{E}(|u_t(x)|^p) < \infty.$$

Stochastic heat equation (Khoshnevisan-Kim'15)

$\partial_t u = \frac{1}{2}\partial_x^2 u + \lambda u \dot{W}$ on $[0,1]$ with Dirichlet BC, $u_0(x) = \sin(\pi x)$

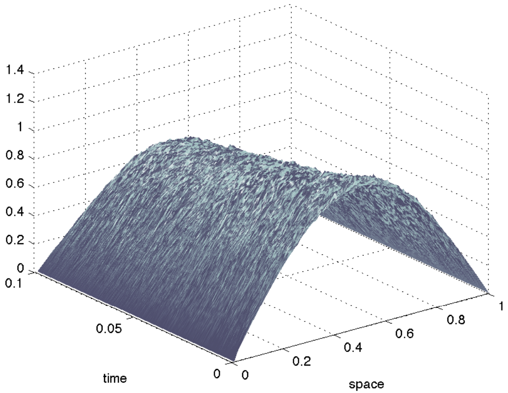


Figure: $\lambda = 0.1$; max. peak ≈ 1.4

Stochastic heat equation (Khoshnevisan-Kim'15)

$\partial_t u = \frac{1}{2}\partial_x^2 u + \lambda u \dot{W}$ on $[0,1]$ with Dirichlet BC, $u_0(x) = \sin(\pi x)$

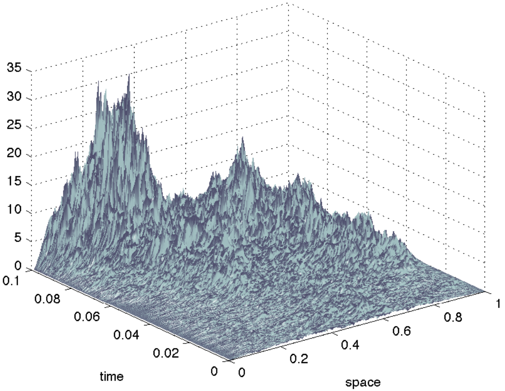


Figure: $\lambda = 2$; max. peak ≈ 35

Stochastic heat equation (Khoshnevisan-Kim'15)

$\partial_t u = \frac{1}{2}\partial_x^2 u + \lambda u \dot{W}$ on $[0,1]$ with Dirichlet BC, $u_0(x) = \sin(\pi x)$

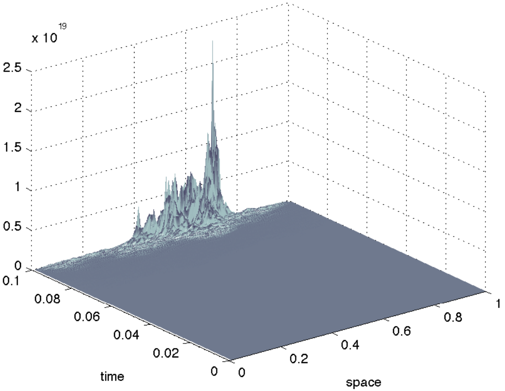


Figure: $\lambda = 5$; max. peak $\approx 2.5 \times 10^{19}$

Intermittency for SPDEs

- Intermittency phenomenon : the solution $u_t(x)$ to the parabolic Anderson model develops high peaks on small x-intervals when t increases.
- Mathematical definition : consider the pth moment Lyapunov exponent

$$\gamma(p) = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^p).$$

- (Gärtner-Molchanov'90) : u is fully intermittent if for all x

$$p \to \frac{\gamma(p)}{p} \text{ is strictly increasing for all } p \geq 2.$$

(it is always nondecreasing)

- (Foondun-Khoshnevisan'09) : u is weakly intermittent if for all x

$$\gamma(2) > 0 \text{ and } \gamma(p) < \infty \text{ for all } p > 2.$$

- If $\gamma(1) = 0$ then weak intermittency implies full intermittency.
- Moreover, if $u_t(x) \geq 0$ a.s. for all $t > 0$ and x then $\gamma(1) = 0$.

Results for large λ

Theorem (Khoshnevisan -Kim'14)

If $\inf_{x\in(0,1)} f(x) > 0$*, then for all* $t > 0$*,*

$$\frac{t}{2} \le \liminf_{\lambda\to\infty} \lambda^{-2} \log \mathcal{E}_t(\lambda) \quad \textit{and} \quad \limsup_{\lambda\to\infty} \lambda^{-4} \log \mathcal{E}_t(\lambda) \le \frac{t}{4},$$

where $\mathcal{E}_t(\lambda) := \sqrt{\mathrm{E}\|u_t\|^2_{L^2(0,1)}}$.

Theorem (Foondun-Joseph'14)

If there exists $\epsilon \in (0, \frac{1}{2})$ *such that* $\inf_{x\in[\epsilon,1-\epsilon]} f(x) > 0$*, then for all* $t > 0$

$$\liminf_{\lambda\to\infty} \frac{\log\log \inf_{x\in[\epsilon,1-\epsilon]} \mathrm{E}|u_t(x)|^2}{\log\lambda} = \limsup_{\lambda\to\infty} \frac{\log\log \sup_{x\in[0,1]} \mathrm{E}|u_t(x)|^2}{\log\lambda} = 4.$$

These results show that for large λ the solution is intermittent.

Intermittency for λ large and small

Assume the initial condition f is non-negative, bounded and has positive support inside $[0, 1]$.

Theorem (Foondun-Nualart'15)

For all $p \geq 2$, there exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ and $x \in (0, 1)$,

$$-\infty < \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^p) < 0.$$

If there exists $\epsilon \in (0, \frac{1}{2})$ such that $\inf_{x\in[\epsilon,1-\epsilon]} f(x) > 0$, then there exists $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$ and $x \in [\epsilon, 1-\epsilon]$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^p) < \infty.$$

This results shows that for $\lambda < \lambda_0$ the solution is nonintermittent, while for $\lambda > \lambda_1$ the solution is intermittent.

Corollary

$\lambda_0 \leq \lambda_1$. *Moreover, there exist* $\bar{\lambda} \in [\lambda_0, \lambda_1]$ *and* $\tilde{\lambda} \in [\lambda_0, \lambda_1]$ *such that resp.*

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}|u_t(x)|^2 = 0, \quad \text{and} \quad \liminf_{t\to\infty} \frac{1}{t} \log \mathrm{E}|u_t(x)|^2 = 0.$$

Moreover, $\bar{\lambda} \leq \tilde{\lambda}$.

Open problem : is $\bar{\lambda} = \tilde{\lambda}$?

Idea of proof

- Proof of $\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^2) < 0$: It suffices that for some $\beta > 0$

$$\|u\|_{2,\,\beta} := \sup_{t>0} \sup_{x\in\mathbf{R}} e^{\beta t} \mathrm{E}|u_t(x)|^2 < \infty.$$

- We show that for all $\beta \in (0, 2\mu_1)$,

$$\|u\|_{2,\,\beta} \le c_1 + c_2 \lambda^2 \|u\|_{2,\,\beta}.$$

- Proof of $\liminf_{t\to\infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^2) > 0$: It suffices that for some $\beta > 0$

$$I_\beta := \int_0^\infty e^{-\beta t} \inf_{x\in[\epsilon,1-\epsilon]} \mathrm{E}|u_t(x)|^2 \,\mathrm{d}t = \infty.$$

- We show that for all $t \ge t_0$,

$$I_\beta \ge c_3 + c_4 \lambda^2 I_\beta.$$

Extension : The fractional stochastic heat equation

- Fractional stochastic heat equation on the ball $D = B(0, R)$

$$\begin{cases} \partial_t u_t(x) = -(-\Delta)^{\alpha/2} u_t(x) + \lambda \sigma(u_t(x)) \dot{F}(t, x) & x \in D, \quad t > 0, \\ u_t(x) = 0 \qquad x \in D^c, \quad t > 0. \end{cases}$$

- Initial condition : measurable and bounded function $u_0 : D \to \mathbf{R}_+$.
- $-(-\Delta)^{\alpha/2}$, $0 < \alpha \leq 2$: L^2-generator of a symmetric α-stable process killed when exiting D.
- Gaussian noise $\dot{F}(t, x)$: white in time and coloured in space :

$$\mathrm{E}\left(\dot{F}(t, x)\dot{F}(s, y)\right) = \delta_0(t - s) g(x - y),$$

  $g : \mathbf{R}^d \to \mathbf{R}_+$ is a nonnegative definite (generalized) function whose Fourier transform $\hat{g} = \mu$ is a tempered measure.
- $\lambda > 0$ (level of the noise).
- $\sigma : \mathbf{R} \to \mathbf{R}$ globally Lipschitz function

The mild formulation

- Following Walsh'86 the mild solution is the random field $u = \{u_t(x)\}_{t>0, x\in D}$ satisfying

$$u_t(x) = \int_D u_0(y) p_D(t,x,y)\, dy + \lambda \int_D \int_0^t p_D(t-s,\ x,\ y)\sigma(u_s(y)) F(ds, dy), \quad (1)$$

  $p_D(t,x,y)$ denotes the Dirichlet fractional heat kernel on D and the stochastic integral is understood in an extended Itô sense.

- Following Dalang'99, if the spectral measure satisfies that

$$\int_{\mathbf{R}^d} \frac{\mu(d\xi)}{1+|\xi|^\alpha} < \infty, \quad (2)$$

  then there exists a unique random field solution u to equation (1).

- Moreover, for all $p \geq 2$ and $T > 0$,

$$\sup_{t\in[0,T], x\in D} \mathrm{E}|u_t(x)|^p < \infty.$$

Examples of spatial correlations

- The Riesz kernel :
$$g(x) = |x|^{-\beta}, \quad 0 < \beta < d.$$
Since $\hat{g}(\xi) = c|\xi|^{-(d-\beta)}$ condition (2) holds iff $\beta < \alpha$.
- The fractional kernel :
$$g(x) = \prod_{i=1}^{d} |x_i|^{2H_i - 2}, \quad \frac{1}{2} < H_i < 1.$$
Since $\hat{g}(\xi) = c\prod_{i=1}^{d} |\xi_i|^{1-2H_i}$ condition (2) holds iff $\sum_{i=1}^{d} H_i > d - \frac{\alpha}{2}$.
- The Bessel kernel :
$$g(x) = \int_0^\infty y^{\frac{\eta-d}{2}} e^{-y} e^{-\frac{|x|^2}{4y}} dy.$$
Since $\hat{g}(\xi) = c(1 + |\xi|^2)^{-\eta/2}$ condition (2) holds iff $\eta > d - \alpha$.
- The space-time whice noise case $g = \delta_0$since $\hat{g}(\xi) = 1$, (2) is only satisfied when $\alpha > d$, that is, $d = 1$ and $1 < \alpha \leq 2$.

Intermittency for λ large and small

Assume that

$$\ell_\sigma |x| \leq |\sigma(x)| \leq L_\sigma |x|,$$

g locally integrable, positive continuous function on $D \setminus \{0\}$,
f is non-negative, bounded with positive support in the clousure of D.

Theorem (Foondun-Nualart'15, Foondun-Guerngar-Nana'17)

For all $p \geq 2$, there exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ and $x \in D$,

$$-\infty < \limsup_{t \to \infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^p) < 0.$$

Moreover, for all $\epsilon > 0$, then there exists $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$ and $x \in B(0, R - \epsilon)$,

$$0 < \liminf_{t \to \infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^p) < \infty.$$

This results shows that for $\lambda < \lambda_0$ the solution is nonintermittent, while for $\lambda > \lambda_1$ the solution is intermittent.

Neumann BC in Foondun-Nualart'15

$$
\begin{aligned}
\text{SPDE}: &\quad \partial_t u = \partial^2_{xx} u + \lambda\sigma(u)\dot{W}, \quad \lambda > 0, \quad t > 0, \quad 0 < x < 1,\\
\text{IC}: &\quad u_0(x) = f(x)\\
\text{NBC}: &\quad \partial_x u_t(0) = \partial_x u_t(1) = 0.
\end{aligned}
$$

- Mild formulation :

$$u_t(x) = \int_0^1 p_t^N(x,y) f(y)dy + \lambda \int_0^1 \int_0^t p_{t-s}^N(x,y)\sigma(u_s(y))W(ds,dy).$$

where $p_t^N(x,y)$ is the Neumann heat kernel :

$$p_t^N(x,y) = 1 + \sum_{n=1}^{\infty} e^{-\mu_n t}\cos(n\pi x)\cos(n\pi y).$$

Neumann BC : intermittency

Theorem (Foondun-Nualart'15)

Assume there exists $\epsilon \in (0, \frac{1}{2})$ *such that* $\inf_{x\in[\epsilon,1-\epsilon]} f(x) > 0$. *Then for all* $x \in (0,1)$ *and* $t > 0$,

$$\mathrm{E}|u_t(x)|^2 \geq c_1 e^{c_2 \lambda^4 t}.$$

In particular, for all $x \in (0,1)$ *and* $\lambda > 0$,

$$0 < \liminf_{t\to\infty} \frac{1}{t} \log \mathrm{E}(|u_t(x)|^2) < \infty.$$

Therefore, the solution is intermittent for all $\lambda > 0$. We use the following Gronwall's inequality :

Theorem (Foondun-Joseph'14)

Suppose that $f(t)$ *is a non-negative integrable function :*

$$f(t) \geq a + kb \int_0^t \frac{f(s)}{\sqrt{t-s}}\, \mathrm{d}s \quad \textit{for all} \quad k, t > 0$$

where $a, b > 0$. *Then, for all* $t > 0$, $f(t) \geq c_1 e^{c_2 k^2 t}$.

The stochastic heat equation in $\mathbf{R}$

$$\begin{aligned} \text{SPDE}: &\quad \partial_t u = \frac{1}{2}\partial^2_{xx} u + \lambda \sigma(u)\dot{W}, \quad \lambda > 0, \quad x \in \mathbf{R} \\ \text{IC}: &\quad u_0(x) = f(x), \end{aligned}$$

Mild formulation in terms of the $p_t^G(x,y)$ =Gaussian heat kernel= $\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$

Theorem (Foondun-Khoshnevisan'09)

1. *Assume*

$$\inf_{x\in\mathbf{R}} \left| \frac{\sigma(x)}{x} \right| > 0 \quad \textit{and} \quad \inf_{x\in\mathbf{R}} f(x) > 0.$$

Then, for all $\lambda > 0$ the solution is intermittent.

2. *Assume*

$$0 < \inf_{x\in\mathbf{R}} |\sigma(x)| \leq \sup_{x\in\mathbf{R}} |\sigma(x)| < \infty.$$

Then, for all $\lambda > 0$, the solution is nonintermittent.

Multiple Extensions : fractional Laplacian, $\mathbf{R}^d$, $\mathbf{Z}^d$, fractional Gaussian noise in time,...

Precise estimates ? Feynman-Kac's formula

Heat equation with cooling term :

$$\begin{aligned} \text{PDE :} \quad & \partial_t u = \partial^2_{xx} u - K(x)u, \quad x \in \mathbf{R} \\ \text{IC :} \quad & u_0(x) = f(x), \end{aligned}$$

- $K(x)$ =amount of external cooling at x.
- $p^G_t(x,y)$ is the transition density of a standard Brownian motion B.
- Feynman-Kac's formula

$$u_t(x) = \mathrm{E}^x \left(f(B_t) \exp\left(- \int_0^t K(B_s) ds \right) \right),$$

where E^x means the expectation conditionned such that $B_0 = x$.

- Consequence of Itô's formula.

Feynman-Kac's formula for SPDEs

Parabolic Anderson model :

$$\text{SPDE}: \quad \partial_t u = \partial^2_{xx} u + \lambda u \dot{W}, \quad \lambda > 0, x \in \mathbf{R}$$
$$\text{IC}: \quad u_0(x) = 1,$$

- $p^G_t(x,y)$ is the transition density of a standard Brownian motion B.
- By Hu-D.Nualart'09, for all $p \geq 2$,

$$\mathrm{E}|u_t(x)|^p = \mathrm{E}\left(\exp\left(\lambda^2 \sum_{1\leq j<k\leq p} \int_0^t \delta_0(B^j_s - B^k_s)ds\right)\right),$$

  where B^i are p iid copies of the Brownian motion B.
- $\int_0^t \delta_0(B_s)ds =$Brownian local time=time spent by the Brownian motion at 0 during the time interval $[0,t]$.

Exact moment Lyapunov exponents

Parabolic Anderson model :

$$\text{SPDE :} \quad \partial_t u = \frac{1}{2}\partial^2_{xx} u + \lambda u \dot{W}, \quad \lambda > 0, x \in \mathbf{R}$$
$$\text{IC :} \quad u_0(x) = 1,$$

Theorem (Bertini-Cancrini'95 and Chen'15)

For every integer $n = 2, 3, \ldots$ *and* $x \in \mathbf{R}$

$$\gamma(n) = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left(u_t(x)^n\right) = \frac{1}{24} n(n^2 - 1)\lambda^4.$$

In particular $\gamma(2) = \frac{\lambda^4}{4}$.

When $n = 2$ there is a formula for the fractional Laplacian in Foondun-Khoshnevisan'09

Feynman-Kac's formula and open problem

Parabolic Anderson model :

$$\text{SPDE}: \quad \partial_t u = \frac{1}{2}\partial_{xx}^2 u + \lambda u \dot{W}, \quad \lambda > 0, 0 < x < 1$$
$$\text{IC}: \quad u_0(x) = 1,$$
$$\text{DBC}: \quad u_t(0) = u_t(1) = 0.$$

- $p_t(x, y)$ is the transition density of a Brownian motion B killed when exiting the interval.
- By Hu-D.Nualart'09, for all $p \geq 2$,

$$\mathrm{E}|u_t(x)|^p = \mathrm{E}_x^B \left(\exp\left(\lambda^2 \sum_{1 \leq j \neq k \leq p} \int_0^t \delta_0(B_s^j - B_s^k) ds \right) \right),$$

where B^i are p iid copies of the Brownian motion B.

Open problem : exact formulas for $\gamma(n)$, $n = 2, 3, \ldots$?

Conjecture : $\gamma(2) = \frac{\lambda^4}{4} - \pi^2 \ldots$

Partial answer : Moment bounds

Consider the fractional stochastic heat equation on $D = B(0,1)$.

- **Hypothesis 1 :** There exist positive constants c_1, c_2 and $0 < \beta < \alpha \wedge d$ such that for all $x \in \mathbf{R}^d$,
$$c_1|x|^{-\beta} \leq g(x) \leq c_2|x|^{-\beta}.$$
- **Hypothesis 2 :** There exist positive constants ℓ_σ, L_σ such that for all $x \in \mathbf{R}^d$,
$$\ell_\sigma|x| \leq |\sigma(x)| \leq L_\sigma|x|.$$
- **Hypothesis 3 :** There exists $\epsilon \in (0, \frac{1}{2})$ such that
$$\inf_{x \in D_\epsilon} f(x) > 0,$$
where $D_\epsilon = \{y \in \mathbf{R}^d : |y| \leq 1 - \epsilon\}$.

Theorem (Nualart'18)

Assume Hypothesis 3.

a) *If g satisfies Hypothesis 1 and $\sigma(x) = x$, then for all $p \geq 2$ and $\delta > 0$,*

$$\overline{c}_2^p e^{pt(c_2 \lambda^{\frac{2\alpha}{\alpha-\beta}} - \mu_1)} \leq \inf_{x \in D_\epsilon} \mathrm{E}|u_t(x)|^p \leq \sup_{x \in D} \mathrm{E}|u_t(x)|^p \leq \overline{c}_1^p e^{pt(c_1 p^{\frac{\alpha}{\alpha-\beta}} \lambda^{\frac{2\alpha}{\alpha-\beta}} - (1-\delta)\mu_1)}.$$

b) *If $g = \delta_0$ and σ satisfies Hypothesis 2, then for all $p \geq 2$ and $\delta > 0$,*

$$\overline{c}_2^p e^{p(c_2 \lambda^{\frac{2\alpha}{\alpha-1}} - \mu_1)} \leq \inf_{x \in D_\epsilon} \mathrm{E}|u_t(x)|^p \leq \sup_{x \in D} \mathrm{E}|u_t(x)|^p \leq \overline{c}_1^p e^{pt(c_1 z_p^{\frac{2\alpha}{\alpha-1}} \lambda^{\frac{2\alpha}{\alpha-1}} - (1-\delta)\mu_1)}.$$

Upper bounds hold for all $t > 0$ while lower bounds holds for all $t > c(\alpha)\lambda^{-\frac{2\alpha}{\alpha-1}}$. When $\alpha = 2$, lower bounds hold for all $t > 0$.

Moment-type Lyapunov upper and lower exponents in terms of $\lambda > 0$:

Corollary

In case a), for all $\lambda > 0$*,*

$$p\left(c_2\lambda^{\frac{2\alpha}{\alpha-\beta}} - \mu_1\right) \le \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in D_\epsilon} \mathrm{E}|u_t(x)|^p$$
$$\le \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} \mathrm{E}|u_t(x)|^p \le p\left(\tilde{c}_1(p)\lambda^{\frac{2\alpha}{\alpha-\beta}} - (1-\delta)\mu_1\right),$$

$\tilde{c}_1(p) = c_1 p^{\frac{\alpha}{\alpha-\beta}}$.

Similar bounds for case b).

Remarks

- The lower bound when $\alpha = 2$ and F is space-time white noise on $\mathbf{R}_+ \times (0, 1)$ was already obtained by Xie'16.

- This theorem implies that for all $p \geq 2$, $t > 0$ and $x \in D_\epsilon$,

$$\lim_{\lambda \to \infty} \frac{\log \log \mathrm{E}|u_t(x)|^p}{\log \lambda} = \frac{2\alpha}{\alpha - a},$$

  which is known as the excitation index of the solution introduced by Khoshnevisan-Kim'15. This result with $p = 2$ was already obtained by Liu-Tian-Foondun'17 in the case that g is the Riesz kernel and σ satisfies Hypothesis 2.

Proposition 1

For any $\epsilon \in (0, \frac{1}{2})$, there exist $c_1(\epsilon)$, $c_2(\epsilon)$ and $c_3(\epsilon)$ such that for all $x \in D_\epsilon$ and $t > 0$,

$$\int_{D_\epsilon} p_D(t,x,y)dy \geq c_1 e^{-\mu_1 t},$$

for all $x \in D_\epsilon$ and $t > 0$,

$$\int_{D_\epsilon} p_D^2(t,x,y)dy \geq c_2 e^{-2\mu_1 t} t^{-d/\alpha},$$

and if g satisfies Hypothesis 2, then for all $x, w \in D_\epsilon$ and $t > 0$ such that $|x - w| \leq t^\alpha$,

$$\int_{D_\epsilon \times D_\epsilon} p_D(t,x,y) p_D(t,w,z) g(y-z) dydz \geq c_3 e^{-2\mu_1 t} t^{-\beta/\alpha}.$$

Proposition 2

For all $\delta > 0$, there exist $c_1, c_2(\delta) > 0$ such that for all $x, w \in D$ and $t > 0$,

$$\int_D p_D(t,x,y)dy \leq c_1 e^{-\mu_1 t},$$

and

$$\int_{D\times D} p_D(t,x,y)p_D(t,w,z)g(y-z)dydz \leq c_2 e^{-(2-\delta)\mu_1 t} t^{-a/\alpha},$$

where

$$a = \begin{cases} d, & \text{if} \quad g = \delta_0, \\ \beta, & \text{if} \quad g \text{ satisfies Hypothesis 2.} \end{cases}$$

Proposition 3 (Henry'81, Foondun-Liu-Omaba'17)

Let $\rho > 0$ and G a locally integrable function satisfying

$$G(t) \leq c_1 + k \int_0^t (t-s)^{\rho-1} G(s) ds \quad \text{for all} \quad t > 0, \tag{3}$$

for some $c_1, k > 0$. Then there exist $c_2, c_3 > 0$ such that

$$G(t) \leq c_2 e^{c_3 \Gamma(\rho)^{1/\rho} k^{1/\rho} t} \quad \text{for all} \quad t > 0.$$

If instead of (3) the function is non-negative and satisfies

$$G(t) \geq c_1 + k \int_0^t (t-s)^{\rho-1} G(s) ds \quad \text{for all} \quad t > 0,$$

then

$$G(t) \geq c_2 e^{c_3 \Gamma(\rho)^{1/\rho} k^{1/\rho} t} \quad \text{for all} \quad t > \frac{e}{\rho} (\Gamma(\rho) k)^{-1/\rho}.$$

If $\rho = \frac{1}{2}$, the latter lower bound holds for all $t > 0$.

Proof of the lower bound

- Case $g = \delta_0$ and $d = 1$.
- By Jensen's inequality, for any $p \geq 2$,

$$\mathrm{E}|u_t(x)|^p \geq \left(\mathrm{E}|u_t(x)|^2\right)^{p/2}.$$

  Therefore, it suffices to prove the lower bound for $p = 2$.
- Taking the second moment to the mild formulation

$$\mathrm{E}|u_t(x)|^2 = \left(\int_D u_0(y) p_D(t,x,y) dy\right)^2 + \lambda^2 \int_0^t \int_D p_D^2(t-s,x,y) \mathrm{E}|\sigma(u_s(y))|^2 dyds.$$

- By the heat kernel estimates, and Hypotheses 2-3,

$$G_\epsilon(t) \geq c\left(1 + \lambda^2 \int_0^t (t-s)^{-1/\alpha} G_\epsilon(s) ds\right),$$

  where

$$G_\epsilon(t) = e^{2\mu_1 t} \inf_{y \in D_\epsilon} \mathrm{E}|u_s(y)|^2.$$

- Proposition 3 with $\rho = 1 - \frac{1}{\alpha}$ concludes.

Proof of the upper bound

- Case $g = \delta_0$ and $d = 1$.
- Taking the pth moment to the mild formulation and appealing to Burkhölder's and Minkowski's inequalities

$$\mathrm{E}|u_t(x)|^p \leq 2^{p-1}\Big\{ \Big(\int_D u_0(y)p_D(t,x,y)dy\Big)^p + \lambda^p c_p \Big(\int_0^t \int_D p_D^2(t-s,x,y)(\mathrm{E}|\sigma(u_s(y))|^p)^{2/p} dyds\Big)^{p/2}\Big\}.$$

- Since u_0 is bounded, and by the heat kernel estimates, and Hypothesis 2,

$$G(t) \leq c\left(1 + \lambda^2 \int_0^t \frac{G(s)}{(t-s)^{1/\alpha}} ds\right),$$

where

$$G(t) = e^{(2-\delta)\mu_1 t} \sup_{y \in D} (\mathrm{E}|u_s(y)|^p)^{2/p}.$$

- Proposition 3 with $\rho = 1 - \frac{1}{\alpha}$ concludes.

Proof for $d > 1$ and $\sigma(x) = x$

- Consider the Wiener-chaos expansion in $L^2(\Omega)$

$$u_t(x) = \sum_{n\geq 0} v_t^{(n)}(x),$$

where $v_t^{(0)}(x) = \int_D u_0(y) p_D(t,x,y)dy$ and for $n \geq 1$,

$$v_t^{(n)}(x) = \lambda^n \int_{\mathbf{R}_+^n} \int_{D^n} p_D(t-t_n, x, x_n) p_D(t_n - t_{n-1}, x_n, x_{n-1})$$
$$\cdots p_D(t_2 - t_1, x_2, x_1) v_{t_1}^{(0)}(x_1) \mathbf{1}_{\{0<t_1<\cdots<t_n<t\}} F(dt_1, dx_1) \cdots F(dt_n, dx_n).$$

- This means that

$$v_t^{(n)}(x) = \lambda^n I_n(h_n(\cdot, t, x)),$$

where I_n denotes the multiple Wiener integral with respect to F, and

$$h_n(t_1, x_1, ..., t_n, x_n, t, x) = p_D(t-t_n, x, x_n) p_D(t_n - t_{n-1}, x_n, x_{n-1})$$
$$\cdots p_D(t_2 - t_1, x_2, x_1) v_{t_1}^{(0)}(x_1) \mathbf{1}_{\{0<t_1<\cdots<t_n<t\}}.$$

- Therefore,

$$\mathrm{E}|u_t(x)|^2 = |v_t^{(0)}(x)|^2 + \sum_{n\geq 1} \lambda^{2n} n! \|\tilde{h}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes 2}},$$

where $\tilde{h}_n$ denotes the symmetrization of h_n. That is,

$$n!\|\tilde{h}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes 2}} = \int_{0<t_1<\cdots<t_n<t} \int_{D^{2n}} p_D(t-t_n, x, x_n) p_D(t-t_n, x, y_n) g(x_n - y_n)$$
$$\times\, p_D(t_n - t_{n-1}, x_n, x_{n-1}) p_D(t_n - t_{n-1}, x_n, y_{n-1}) g(x_{n-1} - y_{n-1}) \cdots p_D(t_2 - t_1, x_2, x_1)$$
$$\times\, p_D(t_2 - t_1, x_2, y_1) g(x_1 - y_1) |v_{t_1}^{(0)}(x_1)|^2 dx_1 \cdots dx_n dy_1 \cdots dy_n dt_1 \cdots dt_n.$$

- The heat kernels estimates imply

$$c_1 e^{-2\mu_1 t} \int_{0<t_1<\cdots<t_n<t} (t-t_n)^{-\beta/\alpha} \prod_{2=1}^{n} (t_i - t_{i-1})^{-\beta/\alpha} dt_1 \cdots dt_n \leq$$

$$n!\|\tilde{h}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes 2}} \leq c_2 e^{-2(1-\delta)\mu_1 t} \int_{0<t_1<\cdots<t_n<t} (t-t_n)^{-\beta/\alpha} \prod_{2=1}^{n} (t_i - t_{i-1})^{-\beta/\alpha} dt_1 \cdots d$$

Proof for $d > 1$ and $\sigma(x) = x$

- Following Balan-Conus'16, we conclude that

$$c_1 e^{-2\mu_1 t} \left(\sum_{n \geq 0} \lambda^{2n} C_1^n (n!)^{\frac{\beta}{\alpha} - 1} t^{-\frac{n\beta}{\alpha} + n} \right) \leq \mathrm{E}|u_t(x)|^2$$

$$\leq c_2 e^{-2\mu_1 (1-\delta) t} \left(\sum_{n \geq 0} \lambda^{2n} C_2^n (n!)^{\frac{\beta}{\alpha} - 1} t^{-\frac{n\beta}{\alpha} + n} \right),$$

and

$$c_1 \exp\left(C_1 \lambda^{\frac{2\alpha}{\alpha - \beta}} t \right) e^{-2\mu_1 t} \leq \mathrm{E}|u_t(x)|^2 \leq c_2 e^{-2(1-\delta)\mu_1 t} \exp\left(C_2 \lambda^{\frac{2\alpha}{\alpha - \beta}} t \right)$$

- Use Minkowski's inequality and the equivalence of the L^p-norms in a fixed chaos

$$\|u_t(x)\|_p \leq \sum_{n \geq 0} \|I_n(h_n(\cdot, t, x))\|_p \leq \sum_{n \geq 0} (p-1)^{n/2} \left(n! \|\tilde{h}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes 2}} \right)^{1/2}$$