Gelfand-Shilov smoothing effect for the Landau equation

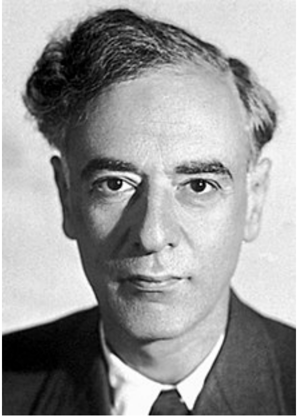
Nicolas Lerner (Sorbonne Université)

after a joint work with

K. Pravda-Starov, Y. Morimoto & C.-J. Xu

Non Standard Diffusions in Fluids, Kinetic Equations and Probability

December 10 – 14, 2018, CIRM



Lev Davidovitch LANDAU (1908–1968)

1. Introduction

Landau Equation and Hypoellipticity

1. Introduction

Landau Equation and Hypoellipticity

The Landau kinetic equation, derived in 1936 by the Russian theoretical physicist Lev Davidovitch Landau reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_L(f, f), \\ f|_{t=0} = f_0, \end{cases} \tag{1}$$

1. Introduction

Landau Equation and Hypoellipticity

The Landau kinetic equation, derived in 1936 by the Russian theoretical physicist Lev Davidovitch Landau reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_L(f, f), \\ f|_{t=0} = f_0, \end{cases} \tag{1}$$

where $f = f(t, x, v) \geq 0$ stands for the density distribution of particles at time t, having position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$.

1. Introduction

Landau Equation and Hypoellipticity

The Landau kinetic equation, derived in 1936 by the Russian theoretical physicist Lev Davidovitch Landau reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_L(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{1}$$

where $f = f(t,x,v) \geq 0$ stands for the density distribution of particles at time t, having position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The term $Q_L(f,f)$ corresponds to the Landau collision operator associated to the bilinear operator

$$Q_L(g,f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^3} a(v - v_*)\{g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*)f(v)\} dv_* \Big),$$

where $a = (a_{i,j})_{1 \leq i,j \leq 3}$ stands for the non-negative symmetric matrix

1. Introduction

Landau Equation and Hypoellipticity

The Landau kinetic equation, derived in 1936 by the Russian theoretical physicist Lev Davidovitch Landau reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q_L(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{1}$$

where $f = f(t,x,v) \geq 0$ stands for the density distribution of particles at time t, having position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The term $Q_L(f,f)$ corresponds to the Landau collision operator associated to the bilinear operator

$$Q_L(g,f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^3} a(v-v_*)\{g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)\} dv_* \Big),$$

where $a = (a_{i,j})_{1 \leq i,j \leq 3}$ stands for the non-negative symmetric matrix

$$a(v) = |v|^{\gamma+2}\big(\mathrm{Id} - \frac{v \otimes v}{|v|^2}\big) = |v|^{\gamma+2}\Pi_{v^\perp} \in \mathcal{M}_3(\mathbb{R}), \quad -3 < \gamma < +\infty.$$

More specifically, we study the Landau equation with Maxwellian molecules in a close to equilibrium framework.

More specifically, we study the Landau equation with Maxwellian molecules in a close to equilibrium framework. The Maxwellian molecules situation corresponds to the case when the parameter $\gamma = 0$ in the cross section.

More specifically, we study the Landau equation with Maxwellian molecules in a close to equilibrium framework. The Maxwellian molecules situation corresponds to the case when the parameter $\gamma = 0$ in the cross section. We consider the fluctuation

$$f = \mu + \sqrt{\mu} g,$$

around the Maxwellian equilibrium distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}. \tag{2}$$

More specifically, we study the Landau equation with Maxwellian molecules in a close to equilibrium framework. The Maxwellian molecules situation corresponds to the case when the parameter $\gamma = 0$ in the cross section. We consider the fluctuation

$$f = \mu + \sqrt{\mu} g,$$

around the Maxwellian equilibrium distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}. \tag{2}$$

This distribution is a stationary solution for the Landau equation since it depends only on v and we have $Q_L(\mu, \mu) = 0$.

More specifically, we study the Landau equation with Maxwellian molecules in a close to equilibrium framework. The Maxwellian molecules situation corresponds to the case when the parameter $\gamma = 0$ in the cross section. We consider the fluctuation

$$f = \mu + \sqrt{\mu} g,$$

around the Maxwellian equilibrium distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}. \tag{2}$$

This distribution is a stationary solution for the Landau equation since it depends only on v and we have $Q_L(\mu, \mu) = 0$. We consider the linearized Landau operator around this equilibrium distribution given by

$$\mathscr{L}_L g = -\mu^{-1/2} Q_L(\mu, \mu^{1/2} g) - \mu^{-1/2} Q_L(\mu^{1/2} g, \mu). \tag{3}$$

$$\mathscr{L}_L g = -\mu^{-1/2} Q_L(\mu, \mu^{1/2} g) - \mu^{-1/2} Q_L(\mu^{1/2} g, \mu),$$

$$Q_L(g,f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^3} a(v - v_*)\{g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)\} dv_* \Big),$$

$$a(v) = |v|^{\gamma+2}\big(\mathrm{Id} - \frac{v \otimes v}{|v|^2}\big) = |v|^{\gamma+2}\Pi_{v^\perp} \in \mathcal{M}_3(\mathbb{R}), \quad -3 < \gamma < +\infty.$$

$$\mathscr{L}_L g = -\mu^{-1/2} Q_L(\mu, \mu^{1/2} g) - \mu^{-1/2} Q_L(\mu^{1/2} g, \mu),$$

$$Q_L(g, f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^3} a(v - v_*)\{g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)\} dv_* \Big),$$

$$a(v) = |v|^{\gamma+2}(\mathrm{Id} - \frac{v \otimes v}{|v|^2}) = |v|^{\gamma+2}\Pi_{v^\perp} \in \mathcal{M}_3(\mathbb{R}), \quad -3 < \gamma < +\infty.$$

The original Landau equation (1) is then reduced to the Cauchy problem for the fluctuation,

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{4}$$

$$\mathscr{L}_L g = -\mu^{-1/2} Q_L(\mu, \mu^{1/2} g) - \mu^{-1/2} Q_L(\mu^{1/2} g, \mu),$$

$$Q_L(g, f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^3} a(v - v_*)\{g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)\} dv_* \Big),$$

$$a(v) = |v|^{\gamma+2}\big(\mathrm{Id} - \frac{v \otimes v}{|v|^2}\big) = |v|^{\gamma+2} \Pi_{v^\perp} \in \mathcal{M}_3(\mathbb{R}), \quad -3 < \gamma < +\infty.$$

The original Landau equation (1) is then reduced to the Cauchy problem for the fluctuation,

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{4}$$

with

$$\Gamma_L(g, f) = \mu^{-1/2} Q_L(\sqrt{\mu} g, \sqrt{\mu} f). \tag{5}$$

The regularity issues for solutions of the Landau Equation were already extensively studied.
1998, C. Villani, Homogeneous case.

The regularity issues for solutions of the Landau Equation were already extensively studied.
1998, C. Villani, Homogeneous case.
2006, C. Mouhot & L. Neumann, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus.*
2008, Y. Guo & R. Strain, *Exponential decay for soft potentials near Maxwellian.*

The regularity issues for solutions of the Landau Equation were already extensively studied.
1998, C. Villani, Homogeneous case.
2006, C. Mouhot & L. Neumann, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus.*
2008, Y. Guo & R. Strain, *Exponential decay for soft potentials near Maxwellian.*
2009, Y. Chen & L. Desvillettes, *Smoothing effects for solutions of Landau Equation. The authors prove that any classical solution becomes immediately smooth with respect to all variables*
2014, S. Liu & X. Ma, *Regularizing effects for solutions of Landau equation.*
2015, A. Bobylev, I. Gamba & I. Potapenko, *On some properties of the Landau kinetic equation.*
2016, K. Carrapatoso, I. Tristani & K.-C.Wu, *Cauchy problem and exponential stability for the inhomogeneous Landau equation.*
2017, K. Carrapatoso, S. Mischler, *Landau equation for very soft and Coulomb potentials near Maxwellians.*

Our goal:

Our goal:

Use some explicit expression in terms of special functions

Our goal:

Use some explicit expression in terms of special functions to calculate nonetheless the linearized part but also the quadratic part

Our goal:

Use some explicit expression in terms of special functions to calculate nonetheless the linearized part but also the quadratic part and show the existence of very regular solutions also with fast decay in the v variable.

We must study the hypoellipticity properties of the operator

$$\mathcal{P} = \partial_t + v \cdot \nabla_x + \mathscr{L}_L.$$

We must study the hypoellipticity properties of the operator

$$\mathcal{P} = \partial_t + v \cdot \nabla_x + \mathscr{L}_L.$$

That operator has a very particular structure, closely related to the Kolmogorov operator

$$\mathcal{K} = \partial_t + v \cdot \nabla_x - \Delta_v.$$

We must study the hypoellipticity properties of the operator

$$\mathcal{P} = \partial_t + v \cdot \nabla_x + \mathscr{L}_L.$$

That operator has a very particular structure, closely related to the Kolmogorov operator

$$\mathcal{K} = \partial_t + v \cdot \nabla_x - \Delta_v.$$

For $\mathcal{K}$ as well as for $\mathcal{P}$, some ellipticity properties are easy to get in the velocity variables v and the main question is related to the understanding of the mechanism leading to regularity in the space variable x.

We must study the hypoellipticity properties of the operator

$$\mathcal{P} = \partial_t + v \cdot \nabla_x + \mathscr{L}_L.$$

That operator has a very particular structure, closely related to the Kolmogorov operator

$$\mathcal{K} = \partial_t + v \cdot \nabla_x - \Delta_v.$$

For $\mathcal{K}$ as well as for $\mathcal{P}$, some ellipticity properties are easy to get in the velocity variables v and the main question is related to the understanding of the mechanism leading to regularity in the space variable x. In the model-case of the Kolmogorov operator, we notice that the bracket identities

$$\left[\partial_{v_j}, \partial_t + v \cdot \nabla_x\right] = \partial_{x_j},$$

show that the missing derivatives ∇_x are in fact obtained as brackets of the transport part

$$X_0 = \partial_t + v \cdot \nabla_x,$$

with vector fields $(X_j)_{1 \le j \le N}$ such that $-\Delta_v = \sum_{1 \le j \le N} X_j^* X_j$, here $X_j = \partial_{v_j}$.

Our scales of regularity and the result

Our scales of regularity and the result

The Gevrey space $G^{\mu}(\mathbb{R}^d)$ is defined as the space of $f \in C^{\infty}(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$|\widehat{f}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/\mu}}, \quad |\xi| \geq 1.$$

G^1: analytic functions $\subset G^s$ for $s > 1$ which contains some C_c^{∞} functions $\subset C^{\infty}$ functions.

Our scales of regularity and the result

The Gevrey space $G^\mu(\mathbb{R}^d)$ is defined as the space of $f \in C^\infty(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$|\widehat{f}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/\mu}}, \quad |\xi| \geq 1.$$

G^1: analytic functions $\subset G^s$ for $s > 1$ which contains some C_c^∞ functions $\subset C^\infty$ functions.

The Gelfand-Shilov space $S_\nu^\mu(\mathbb{R}^d)$ is defined as the space of Schwartz functions $f \in \mathscr{S}(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$\underbrace{|f(v)| \leq Ce^{-\varepsilon|v|^{1/\nu}}, |v| \geq 1,}_{\text{exponential decay}} \qquad \underbrace{|\widehat{f}(\xi)| \leq Ce^{-\varepsilon|\xi|^{1/\mu}}, |\xi| \geq 1.}_{\text{Gevrey regularity of index } \mu} \tag{6}$$

Our scales of regularity and the result

The Gevrey space $G^{\mu}(\mathbb{R}^d)$ is defined as the space of $f \in C^{\infty}(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$|\widehat{f}(\xi)| \le Ce^{-\varepsilon|\xi|^{1/\mu}}, \quad |\xi| \ge 1.$$

G^1: analytic functions $\subset G^s$ for $s > 1$ which contains some C_c^{∞} functions $\subset C^{\infty}$ functions.

The Gelfand-Shilov space $S^{\mu}_{\nu}(\mathbb{R}^d)$ is defined as the space of Schwartz functions $f \in \mathscr{S}(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$\underbrace{|f(v)| \le Ce^{-\varepsilon|v|^{1/\nu}}, |v| \ge 1,}_{\text{exponential decay}} \qquad \underbrace{|\widehat{f}(\xi)| \le Ce^{-\varepsilon|\xi|^{1/\mu}}, |\xi| \ge 1.}_{\text{Gevrey regularity of index } \mu} \tag{6}$$

In particular, Hermite functions (cf. (32)) belong to the symmetric Gelfand-Shilov space $S^{1/2}_{1/2}(\mathbb{R}^d)$. The symmetric Gelfand-Shilov spaces $S^{\mu}_{\mu}(\mathbb{R}^d)$, with $\mu \ge 1/2$, can be characterized through the decomposition into the Hermite basis $(\Psi_{\alpha})_{\alpha\in\mathbb{N}^d}$:

Our scales of regularity and the result

The Gevrey space $G^\mu(\mathbb{R}^d)$ is defined as the space of $f \in C^\infty(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$|\widehat{f}(\xi)| \le Ce^{-\varepsilon|\xi|^{1/\mu}}, \quad |\xi| \ge 1.$$

G^1: analytic functions $\subset G^s$ for $s > 1$ which contains some C_c^∞ functions $\subset C^\infty$ functions.

The Gelfand-Shilov space $S_\nu^\mu(\mathbb{R}^d)$ is defined as the space of Schwartz functions $f \in \mathscr{S}(\mathbb{R}^d)$ such that $\exists C > 0, \varepsilon > 0$,

$$\underbrace{|f(v)| \le Ce^{-\varepsilon|v|^{1/\nu}}, |v| \ge 1,}_{\text{exponential decay}} \qquad \underbrace{|\widehat{f}(\xi)| \le Ce^{-\varepsilon|\xi|^{1/\mu}}, |\xi| \ge 1.}_{\text{Gevrey regularity of index } \mu} \tag{6}$$

In particular, Hermite functions (cf. (32)) belong to the symmetric Gelfand-Shilov space $S_{1/2}^{1/2}(\mathbb{R}^d)$. The symmetric Gelfand-Shilov spaces $S_\mu^\mu(\mathbb{R}^d)$, with $\mu \ge 1/2$, can be characterized through the decomposition into the Hermite basis $(\Psi_\alpha)_{\alpha\in\mathbb{N}^d}$:

$$f \in S_\mu^\mu(\mathbb{R}^d) \Leftrightarrow \exists \varepsilon_0 > 0,\ \|e^{\varepsilon_0 \mathcal{H}^{\frac{1}{2\mu}}} f\|_{L^2} < +\infty$$

where $\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$ is the d-dimensional Harmonic Oscillator.

Our main result shows that the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{7}$$

Our main result shows that the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{7}$$

is globally well-posed for sufficiently small initial data in the Sobolev space $H^{(r,0)}(\mathbb{R}^6_{x,v})$ with $r > 3/2$, where for $r_1, r_2 \geq 0$,

$$H^{(r_1,r_2)}(\mathbb{R}^6_{x,v}) = \{u \in L^2(\mathbb{R}^6_{x,v}) : \langle D_x \rangle^{r_1} \langle D_v \rangle^{r_2} u \in L^2(\mathbb{R}^6_{x,v})\},$$

with $\langle D_x \rangle = (1 + |D_x|^2)^{1/2}$.

Our main result shows that the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{7}$$

is globally well-posed for sufficiently small initial data in the Sobolev space $H^{(r,0)}(\mathbb{R}^6_{x,v})$ with $r > 3/2$, where for $r_1, r_2 \geq 0$,

$$H^{(r_1,r_2)}(\mathbb{R}^6_{x,v}) = \{u \in L^2(\mathbb{R}^6_{x,v}) : \langle D_x \rangle^{r_1} \langle D_v \rangle^{r_2} u \in L^2(\mathbb{R}^6_{x,v})\},$$

with $\langle D_x \rangle = (1 + |D_x|^2)^{1/2}$.

We expect also some Gelfand-Shilov regularity in the velocity variable v, that is Gevrey regularity in v and exponential decay in v and Gevrey regularity in x.

Theorem

Let $r > 3/2$ be given. There exists $\varepsilon_0 > 0$ such that for all $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \leq \varepsilon_0,$$

the Cauchy problem (7) admits a global solution which satisfies

$$g \in L^\infty([0,+\infty[, H^{(r,0)}(\mathbb{R}^6_{x,v})), \quad \mathscr{L}_L g \in L^2([0,+\infty[, H^{(r,0)}(\mathbb{R}^6_{x,v}))\,.$$

Theorem

Let $r > 3/2$ be given. There exists $\varepsilon_0 > 0$ such that for all $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \leq \varepsilon_0,$$

the Cauchy problem (7) admits a global solution which satisfies

$$g \in L^\infty([0,+\infty[, H^{(r,0)}(\mathbb{R}^6_{x,v})), \quad \mathscr{L}_L g \in L^2([0,+\infty[, H^{(r,0)}(\mathbb{R}^6_{x,v}))\,.$$

Furthermore, the solution enjoys the following Gelfand-Shilov regularity in v and Gevrey regularity x for all positive t: there exists $C > 1$ such that for any $t > 0$ and any $\alpha, \beta, \gamma \in \mathbb{N}^3$, we have

Theorem

Let $r > 3/2$ be given. There exists $\varepsilon_0 > 0$ such that for all $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \le \varepsilon_0,$$

the Cauchy problem (7) *admits a global solution which satisfies*

$$g \in L^\infty([0,+\infty[, H^{(r,0)}(\mathbb{R}^6_{x,v})), \quad \mathscr{L}_L g \in L^2([0,+\infty[, H^{(r,0)}(\mathbb{R}^6_{x,v})).$$

Furthermore, the solution enjoys the following Gelfand-Shilov regularity in v and Gevrey regularity x for all positive t: there exists $C > 1$ such that for any $t > 0$ and any $\alpha, \beta, \gamma \in \mathbb{N}^3$, we have

$$\| \underbrace{v^\alpha \partial_v^\beta}_{v} \underbrace{\partial_x^\gamma}_{x} g(t)\|_{L^\infty(\mathbb{R}^6_{x,v})} \le \max\left(\frac{1}{t^{\frac{3}{2}(|\alpha|+|\beta|+|\gamma|)+13}}, 1\right) C^{|\alpha|+|\beta|+|\gamma|+1}$$

$$\times \underbrace{(\alpha!)^{\frac{3}{2}}(\beta!)^{\frac{3}{2}}}_{\text{Gelfand-Shilov } S^{3/2}_{3/2} \text{ in } v} \underbrace{(\gamma!)^{\frac{3}{2}}}_{\text{Gevrey } 3/2 \text{ in } x} \|g_0\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}.$$

Let us note that this result implies that the Cauchy problem (7) enjoys, locally in time, the same regularizing effect as the following evolution equation

$$\begin{cases} \partial_t g + \left(\sqrt{\mathcal{H}_v} + \langle D_x \rangle\right)^{\frac{2}{3}} g = 0, \\ g|_{t=0} = g_0 \in L^2(\mathbb{R}^6_{x,v}), \end{cases} \tag{8}$$

where $\mathcal{H}_v = -\Delta_v + \frac{|v|^2}{4}$ is the Harmonic Oscillator (cf. (32)) in the velocity variables.

Let us note that this result implies that the Cauchy problem (7) enjoys, locally in time, the same regularizing effect as the following evolution equation

$$\begin{cases} \partial_t g + \left(\sqrt{\mathcal{H}_v} + \langle D_x \rangle\right)^{\frac{2}{3}} g = 0, \\ g|_{t=0} = g_0 \in L^2(\mathbb{R}^6_{x,v}), \end{cases} \tag{8}$$

where $\mathcal{H}_v = -\Delta_v + \frac{|v|^2}{4}$ is the Harmonic Oscillator (cf. (32)) in the velocity variables. By using the functional calculus, similarly to the Cauchy problem for the fractional heat equation, the solution of (8) is given by

$$g(t) = e^{-t\left(\sqrt{\mathcal{H}_v} + \langle D_x \rangle\right)^{2/3}} g_0 \in G^{3/2}(\mathbb{R}^3_x; S^{3/2}_{3/2}(\mathbb{R}^3_v)), \quad \text{for all } t > 0,$$

Let us note that this result implies that the Cauchy problem (7) enjoys, locally in time, the same regularizing effect as the following evolution equation

$$\begin{cases} \partial_t g + \left(\sqrt{\mathcal{H}_v} + \langle D_x \rangle\right)^{\frac{2}{3}} g = 0, \\ g|_{t=0} = g_0 \in L^2(\mathbb{R}^6_{x,v}), \end{cases} \tag{8}$$

where $\mathcal{H}_v = -\Delta_v + \frac{|v|^2}{4}$ is the Harmonic Oscillator (cf. (32)) in the velocity variables. By using the functional calculus, similarly to the Cauchy problem for the fractional heat equation, the solution of (8) is given by

$$g(t) = e^{-t\left(\sqrt{\mathcal{H}_v}+\langle D_x \rangle\right)^{2/3}} g_0 \in G^{3/2}(\mathbb{R}^3_x; S^{3/2}_{3/2}(\mathbb{R}^3_v)), \text{ for all } t > 0,$$

where $G^{3/2}$ is the Gevrey function space and $S^{3/2}_{3/2}$ is the Gelfand-Shilov space.

2. Properties of the Landau Equation

On the linearized Landau operator, a special functions approach

With $\mu = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$ standing for the Maxwellian solution, we have

$$\mathscr{L}_L g = -\mu^{-1/2} Q_L(\mu, \mu^{1/2} g) - \mu^{-1/2} Q_L(\mu^{1/2} g, \mu).$$

This is an unbounded symmetric non-negative operator on $L^2(\mathbb{R}^3_v)$, with kernel

$$\mathcal{N} = \mathrm{Span}\Big\{ \underbrace{\Psi_0}_{\mu^{1/2}}, \Psi_{e_1}, \Psi_{e_2}, \Psi_{e_3}, \sum_{j=1}^{3} \Psi_{2e_j} \Big\}, \quad (\Psi_\alpha) \text{ are the Hermite functions}$$

In the case of Maxwellian molecules, the linearized Landau operator may be computed explicitly and we have

$$\mathscr{L}_L = 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big) - \Delta_{\mathbb{S}^2} + \Big[\Delta_{\mathbb{S}^2} - 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\Big]\mathbb{P}_1 + \Big[-\Delta_{\mathbb{S}^2} - 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\Big]\mathbb{P}_2,$$

In the case of Maxwellian molecules, the linearized Landau operator may be computed explicitly and we have

$$\mathscr{L}_L = 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big) - \Delta_{\mathbb{S}^2} + \Big[\Delta_{\mathbb{S}^2} - 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\Big]\mathbb{P}_1 + \Big[-\Delta_{\mathbb{S}^2} - 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\Big]\mathbb{P}_2,$$

where $\mathbb{P}_k = \sum_{|\alpha|=k} \mathbb{P}_\alpha$ denote the orthogonal projection onto the Hermite basis, whereas $\Delta_{\mathbb{S}^2}$ stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$.

In the case of Maxwellian molecules, the linearized Landau operator may be computed explicitly and we have

$$\mathscr{L}_L = 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big) - \Delta_{\mathbb{S}^2} + \Big[\Delta_{\mathbb{S}^2} - 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\Big]\mathbb{P}_1 + \Big[-\Delta_{\mathbb{S}^2} - 2\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\Big]\mathbb{P}_2,$$

where $\mathbb{P}_k = \sum_{|\alpha|=k} \mathbb{P}_\alpha$ denote the orthogonal projection onto the Hermite basis, whereas $\Delta_{\mathbb{S}^2}$ stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$.

Note (cf. (33)) that

$$\Delta_{\mathbb{S}^2} = \frac{1}{2} \sum_{1 \le j,k \le 3} \big(v_j \partial_{v_k} - v_k \partial_{v_j}\big)^2.$$

We set for any $n, l \geq 0$, $-l \leq m \leq l$,

$$\varphi_{n,l,m}(v) = 2^{-3/4}\Big(\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\Big)^{1/2}\Big(\frac{|v|}{\sqrt{2}}\Big)^{l} \underbrace{L_n^{l+\frac{1}{2}}}_{\substack{\text{Laguerre}\\ \text{polynomials}\\ \text{(cf. (35))}}}\Big(\frac{|v|^2}{2}\Big)e^{-\frac{|v|^2}{4}} \underbrace{Y_l^m}_{\substack{\text{spherical}\\ \text{harmonics}}}\Big(\frac{v}{|v|}\Big),$$

We set for any $n, l \geq 0$, $-l \leq m \leq l$,

$$\varphi_{n,l,m}(v) = 2^{-3/4}\Big(\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\Big)^{1/2}\Big(\frac{|v|}{\sqrt{2}}\Big)^{l} \underbrace{L_n^{l+\frac{1}{2}}}_{\substack{\text{Laguerre} \\ \text{polynomials} \\ \text{(cf. (35))}}}\Big(\frac{|v|^2}{2}\Big)e^{-\frac{|v|^2}{4}} \underbrace{Y_l^m}_{\substack{\text{spherical} \\ \text{harmonics}}}\Big(\frac{v}{|v|}\Big),$$

The family $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by eigenvectors of the harmonic oscillator and the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$,

$$\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\varphi_{n,l,m} = (2n+l)\varphi_{n,l,m}, \quad -\Delta_{\mathbb{S}^2}\varphi_{n,l,m} = l(l+1)\varphi_{n,l,m}.$$

We set for any $n, l \geq 0$, $-l \leq m \leq l$,

$$\varphi_{n,l,m}(v) = 2^{-3/4}\Big(\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\Big)^{1/2}\Big(\frac{|v|}{\sqrt{2}}\Big)^{l}\underbrace{L_n^{l+\frac{1}{2}}}_{\substack{\text{Laguerre}\\ \text{polynomials}\\ \text{(cf. (35))}}}\Big(\frac{|v|^2}{2}\Big)e^{-\frac{|v|^2}{4}}\underbrace{Y_l^m}_{\substack{\text{spherical}\\ \text{harmonics}}}\Big(\frac{v}{|v|}\Big),$$

The family $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by eigenvectors of the harmonic oscillator and the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$,

$$\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\varphi_{n,l,m} = (2n+l)\varphi_{n,l,m}, \quad -\Delta_{\mathbb{S}^2}\varphi_{n,l,m} = l(l+1)\varphi_{n,l,m}.$$

The linearized Landau operator $\mathscr{L}_L$ is diagonal in that basis and satisfies

$$\boxed{\mathscr{L}_L\varphi_{n,l,m} = \lambda_L(n,l,m)\varphi_{n,l,m}. \quad n,l \geq 0,\ -l \leq m \leq l,}$$

We set for any $n, l \geq 0$, $-l \leq m \leq l$,

$$\varphi_{n,l,m}(v) = 2^{-3/4}\Big(\frac{2n!}{\Gamma(n+l+\frac{3}{2})}\Big)^{1/2}\Big(\frac{|v|}{\sqrt{2}}\Big)^{l} \underbrace{L_n^{l+\frac{1}{2}}}_{\text{Laguerre polynomials (cf. (35))}}\Big(\frac{|v|^2}{2}\Big)e^{-\frac{|v|^2}{4}} \underbrace{Y_l^m}_{\text{spherical harmonics}}\Big(\frac{v}{|v|}\Big),$$

The family $(\varphi_{n,l,m})_{n,l\geq 0,|m|\leq l}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by eigenvectors of the harmonic oscillator and the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$,

$$\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\Big)\varphi_{n,l,m} = (2n+l)\varphi_{n,l,m}, \quad -\Delta_{\mathbb{S}^2}\varphi_{n,l,m} = l(l+1)\varphi_{n,l,m}.$$

The linearized Landau operator $\mathscr{L}_L$ is diagonal in that basis and satisfies

$$\boxed{\mathscr{L}_L\varphi_{n,l,m} = \lambda_L(n,l,m)\varphi_{n,l,m}. \quad n, l \geq 0, \ -l \leq m \leq l,}$$

where $\lambda_L(0,0,0) = \lambda_L(0,1,0) = \lambda_L(0,1,\pm 1) = \lambda_L(1,0,0) = 0$, $\lambda_L(0,2,m) = 12$,

$$\lambda_L(n,l,m) = 2(2n+l) + l(l+1), \quad 2n+l > 2.$$

We shall use the following coercive estimates, where **P** is the projection on the kernel $\mathcal{N}$,

$$(\mathscr{L}_L f, f)_{L^2(\mathbb{R}^3_v)} \approx \|\mathcal{H}^{\frac{1}{2}}(1-\mathbf{P})f\|^2_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}(1-\mathbf{P})f\|^2_{L^2(\mathbb{R}^3_v)},$$

where the square root of the Laplace-Beltrami operator is defined by Functional Calculus,

$$(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} f = \sum_{n,l\geq 0, |m|\leq l} \sqrt{l(l+1)}(f, \varphi_{n,l,m})_{L^2(\mathbb{R}^3_v)} \varphi_{n,l,m}.$$

Trilinear estimates for the collision term

Proposition 1.

Trilinear estimates for the collision term

Proposition 1. *Let* $\mathbf{S}_2 = \sum_{0 \le k \le 2} \mathbb{P}_k$, *where* $\mathbb{P}_k$ *denote the orthogonal projections onto the Hermite basis.*

Trilinear estimates for the collision term

Proposition 1. *Let* $\mathbf{S}_2 = \sum_{0\le k\le 2} \mathbb{P}_k$, *where* $\mathbb{P}_k$ *denote the orthogonal projections onto the Hermite basis. Then, there exists a positive constant* $c > 0$ *such that for all* $f, g, h \in \mathscr{S}(\mathbb{R}^3_v)$,

$$\left|\Big(\Gamma_L(g,f),h\Big)_{L^2(\mathbb{R}^3_v)}\right| \le c\|\mathbf{S}_2 g\|_{L^2(\mathbb{R}^3_v)}\Big(\|\mathcal{H}^{\frac12}f\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac12}f\|_{L^2(\mathbb{R}^3_v)}\Big) \times \Big(\|\mathcal{H}^{\frac12}h\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac12}h\|_{L^2(\mathbb{R}^3_v)}\Big),$$

Trilinear estimates for the collision term

Proposition 1. *Let* $\mathbf{S}_2 = \sum_{0 \le k \le 2} \mathbb{P}_k$, *where* $\mathbb{P}_k$ *denote the orthogonal projections onto the Hermite basis. Then, there exists a positive constant* $c > 0$ *such that for all* $f, g, h \in \mathscr{S}(\mathbb{R}^3_v)$,

$$\left|\left(\Gamma_L(g,f),h\right)_{L^2(\mathbb{R}^3_v)}\right| \le c\|\mathbf{S}_2 g\|_{L^2(\mathbb{R}^3_v)}\left(\|\mathcal{H}^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^3_v)}\right) \times \left(\|\mathcal{H}^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3_v)}\right),$$

where $\Gamma_L(g,f) = \mu^{-1/2}Q_L(\sqrt{\mu}g, \sqrt{\mu}f)$.

Trilinear estimates for the collision term

Proposition 1. *Let $\mathbf{S}_2 = \sum_{0 \le k \le 2} \mathbb{P}_k$, where $\mathbb{P}_k$ denote the orthogonal projections onto the Hermite basis. Then, there exists a positive constant $c > 0$ such that for all $f, g, h \in \mathscr{S}(\mathbb{R}^3_v)$,*

$$\left|\Big(\Gamma_L(g,f),h\Big)_{L^2(\mathbb{R}^3_v)}\right| \le c\|\mathbf{S}_2 g\|_{L^2(\mathbb{R}^3_v)}\Big(\|\mathcal{H}^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^3_v)}\Big) \times \Big(\|\mathcal{H}^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3_v)}\Big),$$

where $\Gamma_L(g,f) = \mu^{-1/2}Q_L(\sqrt{\mu}g, \sqrt{\mu}f)$.

We omit the proof, which is long and technical (not so unusual),

Trilinear estimates for the collision term

Proposition 1. *Let $\mathbf{S}_2 = \sum_{0\le k\le 2} \mathbb{P}_k$, where $\mathbb{P}_k$ denote the orthogonal projections onto the Hermite basis. Then, there exists a positive constant $c > 0$ such that for all $f, g, h \in \mathscr{S}(\mathbb{R}^3_v)$,*

$$\left|\Big(\Gamma_L(g,f),h\Big)_{L^2(\mathbb{R}^3_v)}\right| \le c\|\mathbf{S}_2 g\|_{L^2(\mathbb{R}^3_v)}\Big(\|\mathcal{H}^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^3_v)}\Big) \times \Big(\|\mathcal{H}^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3_v)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^3_v)}\Big),$$

where $\Gamma_L(g,f) = \mu^{-1/2}Q_L(\sqrt{\mu}g, \sqrt{\mu}f)$.

We omit the proof, which is long and technical (not so unusual), but we want to point out that the calculations use the basis $(\varphi_{n,l,m})$ displayed above to calculate explicitly the bilinear term $\Gamma_L(g,f)$.

The full Landau equation (and not only the linearized part) can be expressed explicitly as an infinite-dimensional system in the basis $(\varphi_{n,l,m})$.

Proposition 2. *Let $r > 3/2$ be given; there exists a positive constant $C_r > 0$ such that for all $f, g, h \in \mathscr{S}(\mathbb{R}^6_{x,v})$, $t \geq 0$, $0 < \delta \leq 1$,*

$$\Big| \Big(M_\delta(t)\Gamma_L\big(M_\delta(t)^{-1}g, M_\delta(t)^{-1}f\big), h\Big)_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \Big|$$
$$\leq C_r \|\mathbf{S}_2 g\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \Big(\|\mathcal{H}^{\frac{1}{2}} f\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} f\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \Big)$$
$$\times \Big(\|\mathcal{H}^{\frac{1}{2}} h\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} h\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} \Big),$$

Proposition 2. *Let $r > 3/2$ be given; there exists a positive constant $C_r > 0$ such that for all $f, g, h \in \mathscr{S}(\mathbb{R}^6_{x,v})$, $t \geq 0$, $0 < \delta \leq 1$,*

$$\Big|\Big(M_\delta(t)\Gamma_L\big(M_\delta(t)^{-1}g, M_\delta(t)^{-1}f\big), h\Big)_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\Big|$$

$$\leq C_r\|\mathbf{S}_2 g\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\big(\|\mathcal{H}^{\frac{1}{2}}f\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}f\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\big)$$

$$\times\big(\|\mathcal{H}^{\frac{1}{2}}h\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}h\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\big),$$

with the exponential weight

$$M_\delta(t) = \frac{e^{t(\sqrt{\mathcal{H}}+\langle D_x\rangle)^{\frac{2}{3}}}}{1+\delta e^{t(\sqrt{\mathcal{H}}+\langle D_x\rangle)^{\frac{2}{3}}}} = \sum_{k\geq 0}\frac{e^{t(\sqrt{k+\frac{3}{2}}+\langle D_x\rangle)^{\frac{2}{3}}}}{1+\delta e^{t(\sqrt{k+\frac{3}{2}}+\langle D_x\rangle)^{\frac{2}{3}}}}\mathbb{P}_k.$$

Proposition 2. *Let $r > 3/2$ be given; there exists a positive constant $C_r > 0$ such that for all $f, g, h \in \mathscr{S}(\mathbb{R}^6_{x,v})$, $t \geq 0$, $0 < \delta \leq 1$,*

$$\left|\Big(M_\delta(t)\Gamma_L\big(M_\delta(t)^{-1}g, M_\delta(t)^{-1}f\big), h\Big)_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\right|$$

$$\leq C_r \|\mathbf{S}_2 g\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\Big(\|\mathcal{H}^{\frac{1}{2}} f\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} f\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\Big)$$

$$\times \Big(\|\mathcal{H}^{\frac{1}{2}} h\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} h\|_{H^{(r,0)}(\mathbb{R}^6_{x,v})}\Big),$$

with the exponential weight

$$M_\delta(t) = \frac{e^{t(\sqrt{\mathcal{H}}+\langle D_x\rangle)^{\frac{2}{3}}}}{1+\delta e^{t(\sqrt{\mathcal{H}}+\langle D_x\rangle)^{\frac{2}{3}}}} = \sum_{k\geq 0} \frac{e^{t(\sqrt{k+\frac{3}{2}}+\langle D_x\rangle)^{\frac{2}{3}}}}{1+\delta e^{t(\sqrt{k+\frac{3}{2}}+\langle D_x\rangle)^{\frac{2}{3}}}}\mathbb{P}_k.$$

Same remark as for Proposition 1: an intricate proof, but based upon explicit computations, thanks to the special functions approach displayed above.

3. A local existence result

Local existence for a linearized equation

Proposition 3. *Let $r > 3/2$. Then, there exist some positive constants $c_0 > 1$, $\varepsilon_0 > 0$ such that for all $T > 0$, $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, $f \in L^\infty([0,T], H^{(r,0)}(\mathbb{R}^6_{x,v}))$ satisfying $\|\mathbf{S}_2 f\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq \varepsilon_0$, the Cauchy problem*

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(f,g), \\ g|_{t=0} = g_0, \end{cases}$$

3. A local existence result

Local existence for a linearized equation

Proposition 3. *Let $r > 3/2$. Then, there exist some positive constants $c_0 > 1$, $\varepsilon_0 > 0$ such that for all $T > 0$, $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, $f \in L^\infty([0,T], H^{(r,0)}(\mathbb{R}^6_{x,v}))$ satisfying $\|\mathbf{S}_2 f\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \le \varepsilon_0$, the Cauchy problem*

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(f,g), \\ g|_{t=0} = g_0, \end{cases}$$

admits a solution satisfying

$$\|g\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}} g\|_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} g\|_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \le c_0 e^T \|g_0\|_{(r,0)}.$$

---

3. A local existence result

Local existence for a linearized equation

Proposition 3. *Let $r > 3/2$. Then, there exist some positive constants $c_0 > 1$, $\varepsilon_0 > 0$ such that for all $T > 0$, $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, $f \in L^\infty([0,T], H^{(r,0)}(\mathbb{R}^6_{x,v}))$ satisfying $\|\mathbf{S}_2 f\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq \varepsilon_0$, the Cauchy problem*

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(f,g), \\ g|_{t=0} = g_0, \end{cases}$$

admits a solution satisfying

$$\|g\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}} g\|_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} g\|_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq c_0 e^T \|g_0\|_{(r,0)}.$$

A rather straightforward and standard argument: prove an estimate for the adjoint equation (thanks to Proposition 1) and use Hahn-Banach theorem for the existence.

Convergence of approximate solutions

Theorem

Let $r > 3/2$, $T > 0$. Then, there exist some positive constants $c_0 > 1$, $\varepsilon_0 > 0$ such that for all $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$ satisfying $\|g_0\|_{(r,0)} \leq \varepsilon_0$, the Cauchy problem associated to the Landau equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{9}$$

Convergence of approximate solutions

Theorem

Let $r > 3/2$, $T > 0$. Then, there exist some positive constants $c_0 > 1$, $\varepsilon_0 > 0$ such that for all $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$ satisfying $\|g_0\|_{(r,0)} \leq \varepsilon_0$, the Cauchy problem associated to the Landau equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{9}$$

admits a solution satisfying

$$\|g\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}} g\|_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} g\|_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq c_0 \|g_0\|_{(r,0)}.$$

To prove that theorem, thanks to Proposition 3 (establishing existence for the linearized equation), we can define a sequence of solutions $(g_n)_{n\geq 0}$ belonging to $L^\infty([0,T], H^{(r,0)}(\mathbb{R}^6_{x,v}))$ and satisfying the Cauchy problem

To prove that theorem, thanks to Proposition 3 (establishing existence for the linearized equation), we can define a sequence of solutions $(g_n)_{n\geq 0}$ belonging to $L^\infty([0,T], H^{(r,0)}(\mathbb{R}^6_{x,v}))$ and satisfying the Cauchy problem

$$\begin{cases} \partial_t g_{n+1} + v\cdot\nabla_x g_{n+1} + \mathscr{L}_L g_{n+1} = \Gamma_L(g_n, g_{n+1}), \quad n\geq 0, \\ g_{n+1}|_{t=0} = g_0. \end{cases}$$

To prove that theorem, thanks to Proposition 3 (establishing existence for the linearized equation), we can define a sequence of solutions $(g_n)_{n\geq 0}$ belonging to $L^\infty([0,T], H^{(r,0)}(\mathbb{R}^6_{x,v}))$ and satisfying the Cauchy problem

$$\begin{cases} \partial_t g_{n+1} + v\cdot\nabla_x g_{n+1} + \mathscr{L}_L g_{n+1} = \Gamma_L(g_n, g_{n+1}), \quad n\geq 0, \\ g_{n+1}\big|_{t=0} = g_0. \end{cases}$$

We prove the convergence of these solutions towards a solution of the non-linear equation, using essentially the estimates of Proposition 1.

4. Gelfand-Shilov and Gevrey smoothing effects

Hypoellipticity for the linear inhomogeneous Landau equation

We define $\boxed{\mathscr{P} = iv \cdot \xi + \mathscr{L}_L,}$ where ξ is a parameter in $\mathbb{R}^3$.

4. Gelfand-Shilov and Gevrey smoothing effects

Hypoellipticity for the linear inhomogeneous Landau equation

We define $\boxed{\mathscr{P} = iv \cdot \xi + \mathscr{L}_L,}$ where ξ is a parameter in $\mathbb{R}^3$.

Proposition 4. *There exists a real-valued symbol $\tilde{g}(v, \eta, \xi)$ belonging to the symbol class $S(1, dv^2 + d\eta^2)$ uniformly with respect to the parameter $\xi \in \mathbb{R}^3$ and some positive constants $0 < \varepsilon_0 \leq 1$, $c_1, c_2 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $f \in \mathscr{S}(\mathbb{R}^3_v)$, $\xi \in \mathbb{R}^3$,*

$$Re(\mathscr{P}f, (1 - \varepsilon \tilde{g}^w(v, D_v, \xi))f)_{L^2} \geq c_1 \|\mathcal{H}^{\frac{1}{2}} f\|^2_{L^2} + c_1 \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} f\|^2_{L^2}$$
$$+ c_1 \varepsilon \|\langle \xi \rangle^{\frac{1}{3}} f\|^2_{L^2} + c_1 \varepsilon \||v \wedge \xi|^{\frac{1}{3}} f\|^2_{L^2} - c_2 \|f\|^2_{L^2},$$

where $\|\cdot\|_{L^2}$ stands for the $L^2(\mathbb{R}^3_v)$-norm.

4. Gelfand-Shilov and Gevrey smoothing effects

Hypoellipticity for the linear inhomogeneous Landau equation

We define $\boxed{\mathscr{P} = iv \cdot \xi + \mathscr{L}_L,}$ where ξ is a parameter in $\mathbb{R}^3$.

Proposition 4. *There exists a real-valued symbol $\tilde{g}(v, \eta, \xi)$ belonging to the symbol class $S(1, dv^2 + d\eta^2)$ uniformly with respect to the parameter $\xi \in \mathbb{R}^3$ and some positive constants $0 < \varepsilon_0 \leq 1$, $c_1, c_2 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $f \in \mathscr{S}(\mathbb{R}^3_v)$, $\xi \in \mathbb{R}^3$,*

$$Re(\mathscr{P}f, (1 - \varepsilon \tilde{g}^w(v, D_v, \xi))f)_{L^2} \geq c_1 \|\mathcal{H}^{\frac{1}{2}} f\|^2_{L^2} + c_1 \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} f\|^2_{L^2}$$
$$+ c_1 \varepsilon \|\langle \xi \rangle^{\frac{1}{3}} f\|^2_{L^2} + c_1 \varepsilon \||v \wedge \xi|^{\frac{1}{3}} f\|^2_{L^2} - c_2 \|f\|^2_{L^2},$$

where $\|\cdot\|_{L^2}$ stands for the $L^2(\mathbb{R}^3_v)$-norm.

Note. $S(1, dv^2 + d\eta^2)$ stands for the smooth functions whose each derivative is bounded, a^w for the Weyl quantization of a.

4. Gelfand-Shilov and Gevrey smoothing effects

Hypoellipticity for the linear inhomogeneous Landau equation

We define $\boxed{\mathscr{P} = iv \cdot \xi + \mathscr{L}_L,}$ where ξ is a parameter in $\mathbb{R}^3$.

Proposition 4. *There exists a real-valued symbol $\tilde{g}(v, \eta, \xi)$ belonging to the symbol class $S(1, dv^2 + d\eta^2)$ uniformly with respect to the parameter $\xi \in \mathbb{R}^3$ and some positive constants $0 < \varepsilon_0 \leq 1$, $c_1, c_2 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $f \in \mathscr{S}(\mathbb{R}^3_v)$, $\xi \in \mathbb{R}^3$,*

$$Re(\mathscr{P}f, (1 - \varepsilon \tilde{g}^w(v, D_v, \xi))f)_{L^2} \geq c_1 \|\mathcal{H}^{\frac{1}{2}} f\|^2_{L^2} + c_1 \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} f\|^2_{L^2}$$
$$+ c_1 \varepsilon \|\langle \xi \rangle^{\frac{1}{3}} f\|^2_{L^2} + c_1 \varepsilon \||v \wedge \xi|^{\frac{1}{3}} f\|^2_{L^2} - c_2 \|f\|^2_{L^2},$$

where $\|\cdot\|_{L^2}$ stands for the $L^2(\mathbb{R}^3_v)$-norm.

Note. $S(1, dv^2 + d\eta^2)$ stands for the smooth functions whose each derivative is bounded, a^w for the Weyl quantization of a.

Proof. Hmm, turns somewhat technological: we use a lemma due to F. Hérau & K. Pravda-Starov grounded on Wick quantization as used in Lerner's book on Pseudo-Differential Operators.

A priori estimates with exponential weights

Proposition 5. *Let $r > 3/2$, $T > 0$. Then, there exist some positive constants $c, \delta_0, \varepsilon_1 > 0$ such that for all initial data $\|g_0\|_{(r,0)} \leq \varepsilon_1$, the sequence of approximate solutions $(g_n)_{n\geq 0}$ defined above satisfies for all $0 \leq \delta \leq \delta_0$, $n \geq 1$,*

$$\|M_0(\delta t)g_n\|^2_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))}$$
$$+ \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \||v \wedge D_x|^{\frac{1}{3}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))}$$
$$+ \|\langle D_x\rangle^{\frac{1}{3}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq ce^{cT}\|g_0\|^2_{H^{(r,0)}(\mathbb{R}^6_{x,v})},$$

A priori estimates with exponential weights

Proposition 5. *Let $r > 3/2$, $T > 0$. Then, there exist some positive constants $c, \delta_0, \varepsilon_1 > 0$ such that for all initial data $\|g_0\|_{(r,0)} \leq \varepsilon_1$, the sequence of approximate solutions $(g_n)_{n\geq 0}$ defined above satisfies for all $0 \leq \delta \leq \delta_0$, $n \geq 1$,*

$$\|M_0(\delta t)g_n\|^2_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))}$$
$$+ \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \||v \wedge D_x|^{\frac{1}{3}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))}$$
$$+ \|\langle D_x\rangle^{\frac{1}{3}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq ce^{cT}\|g_0\|^2_{H^{(r,0)}(\mathbb{R}^6_{x,v})},$$

where $M_0(t) = e^{t(\sqrt{\mathcal{H}}+\langle D_x\rangle)^{\frac{2}{3}}}$.

A priori estimates with exponential weights

Proposition 5. *Let $r > 3/2$, $T > 0$. Then, there exist some positive constants $c, \delta_0, \varepsilon_1 > 0$ such that for all initial data $\|g_0\|_{(r,0)} \leq \varepsilon_1$, the sequence of approximate solutions $(g_n)_{n\geq 0}$ defined above satisfies for all $0 \leq \delta \leq \delta_0$, $n \geq 1$,*

$$\|M_0(\delta t)g_n\|^2_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))}$$
$$+ \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} + \||v \wedge D_x|^{\frac{1}{3}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))}$$
$$+ \|\langle D_x\rangle^{\frac{1}{3}} M_0(\delta t)g_n\|^2_{L^2([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq ce^{cT}\|g_0\|^2_{H^{(r,0)}(\mathbb{R}^6_{x,v})},$$

where $M_0(t) = e^{t(\sqrt{\mathcal{H}}+\langle D_x\rangle)^{\frac{2}{3}}}$.

Essentially a consequence of Proposition 4. Provides as well the following result.

Theorem

Let $r > 3/2$, $T > 0$. Then, there exist some positive constants $\varepsilon_0 > 0$, $c_0 > 1$, $\delta > 0$ such that for all $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{(r,0)} \leq \varepsilon_0,$$

the Cauchy problem associated to the Landau equation with Maxwellian molecules (4) admits a solution on $[0, T]$ satisfying

$$\left\| \exp\left(\delta t(\sqrt{\mathcal{H}} + \langle D_x\rangle)^{\frac{2}{3}}\right) g\right\|_{L^\infty([0,T],H^{(r,0)}(\mathbb{R}^6_{x,v}))} \leq c_0 \|g_0\|_{(r,0)}.$$

5. Global existence of the solution

Micro-macro energy estimate

We want to establish the global existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases}$$

for a sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r > 3/2$.

5. Global existence of the solution

Micro-macro energy estimate

We want to establish the global existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases}$$

for a sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r > 3/2$. To that end, we first consider the case when $r = 2$ and use the standard micro-macro decomposition

$$g = \mathbf{P}g + (\mathsf{Id} - \mathbf{P})g = g_1 + g_2, \tag{10}$$

where $\mathbf{P}$ stands for the $L^2(\mathbb{R}^3_v)$-orthogonal projection onto the space of collisional invariants

$$\mathcal{N} = \mathrm{Span}\{\mu^{1/2}, v_1\mu^{1/2}, v_2\mu^{1/2}, v_3\mu^{1/2}, |v|^2\mu^{1/2}\}.$$

We want to establish the global existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases}$$

for a sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r > 3/2$.

We want to establish the global existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases}$$

for a sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r > 3/2$.
To that end, we first consider the case when $r = 2$ and use the standard micro-macro decomposition

$$g = \mathbf{P}g + (\mathrm{Id} - \mathbf{P})g = g_1 + g_2. \tag{11}$$

We want to establish the global existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases}$$

for a sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r > 3/2$.
To that end, we first consider the case when $r = 2$ and use the standard micro-macro decomposition

$$g = \mathbf{P}g + (\mathrm{Id} - \mathbf{P})g = g_1 + g_2. \tag{11}$$

We first derive an energy estimate for both the macroscopic part of the solution g_1 and its microscopic part g_2.

We want to establish the global existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases}$$

for a sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r > 3/2$.
To that end, we first consider the case when $r = 2$ and use the standard micro-macro decomposition

$$g = \mathbf{P}g + (\mathrm{Id} - \mathbf{P})g = g_1 + g_2. \tag{11}$$

We first derive an energy estimate for both the macroscopic part of the solution g_1 and its microscopic part g_2. Thanks to this energy estimate, we prove a result of global existence for sufficiently small initial data $g_0 \in H^{(r,0)}(\mathbb{R}^6_{x,v})$, with $r = 2$. By using next the smoothing effect, we establish the result of global existence in the general case when $r > 3/2$.

$$g_1 = \mathbf{P}g = \big(a + b\cdot v + c|v|^2\big)\mu^{1/2}, \quad g_2 = (\mathrm{Id} - \mathbf{P})g$$

$$\begin{cases} \partial_t g + v\cdot\nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{12}$$

$$g_1 = \mathbf{P}g = \left(a + b \cdot v + c|v|^2\right)\mu^{1/2}, \quad g_2 = (\mathrm{Id} - \mathbf{P})g$$

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{12}$$

Proposition 6. Let $r = 2$. There exist some positive constants $\varepsilon_0 > 0$, $C > 0$ such that if $g \in L^\infty([0,T], H^{(2,0)}(\mathbb{R}^6_{x,v}))$ is a solution to the Cauchy problem (12) on $[0,T]$ satisfying $\sup_{0\le t\le T} \mathcal{E}(t) \le \varepsilon_0$, then the solution enjoys the estimate

$$g_1 = \mathbf{P}g = \left(a + b \cdot v + c|v|^2\right)\mu^{1/2}, \quad g_2 = (\mathrm{Id} - \mathbf{P})g$$

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + \mathscr{L}_L g = \Gamma_L(g, g), \\ g|_{t=0} = g_0, \end{cases} \tag{12}$$

Proposition 6. Let $r = 2$. There exist some positive constants $\varepsilon_0 > 0$, $C > 0$ such that if $g \in L^\infty([0, T], H^{(2,0)}(\mathbb{R}^6_{x,v}))$ is a solution to the Cauchy problem (12) on $[0, T]$ satisfying $\sup_{0 \le t \le T} \mathcal{E}(t) \le \varepsilon_0$, then the solution enjoys the estimate

$$\forall 0 \le t \le T, \quad \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \le C\mathcal{E}(0),$$

with the instant energy functional

$$g_1 = \mathbf{P}g = \big(a + b\cdot v + c|v|^2\big)\mu^{1/2}, \quad g_2 = (\mathrm{Id} - \mathbf{P})g$$

$$\begin{cases} \partial_t g + v\cdot\nabla_x g + \mathscr{L}_L g = \Gamma_L(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{12}$$

Proposition 6. Let $r = 2$. There exist some positive constants $\varepsilon_0 > 0$, $C > 0$ such that if $g \in L^\infty([0,T], H^{(2,0)}(\mathbb{R}^6_{x,v}))$ is a solution to the Cauchy problem (12) on $[0,T]$ satisfying $\sup_{0\le t\le T}\mathcal{E}(t) \le \varepsilon_0$, then the solution enjoys the estimate

$$\forall 0 \le t \le T, \quad \mathcal{E}(t) + \int_0^t \mathcal{D}(\tau)d\tau \le C\mathcal{E}(0),$$

with the instant energy functional

$$\mathcal{E}(t) = \|g(t)\|^2_{(2,0)} \sim \|(a,b,c)(t)\|^2_{H^2(\mathbb{R}^3)} + \|g_2(t)\|^2_{(2,0)}$$

and the total dissipation rate

$$\mathcal{D}(t) = \|\nabla_x(a,b,c)(t)\|^2_{H^1(\mathbb{R}^3)} + \|\mathcal{H}^{\frac{1}{2}}g_2(t)\|^2_{(2,0)} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}g_2(t)\|^2_{(2,0)}.$$

Global existence

Proposition 7. Let $r = 2$. Then, there exist some positive constants $c_1 > 1$, $\varepsilon_1 > 0$ such that for all $g_0 \in H^{(2,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{(2,0)} \leq \varepsilon_1,$$

the Cauchy problem admits a global solution satisfying

Global existence

Proposition 7. Let $r = 2$. Then, there exist some positive constants $c_1 > 1$, $\varepsilon_1 > 0$ such that for all $g_0 \in H^{(2,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{(2,0)} \leq \varepsilon_1,$$

the Cauchy problem admits a global solution satisfying

$$\|g\|_{L^\infty([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}}(1-\mathbf{P})g\|_{L^2([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}(1-\mathbf{P})g\|_{L^2([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))} \leq c_1\|g_0\|_{(2,0)},$$

Global existence

Proposition 7. Let $r = 2$. Then, there exist some positive constants $c_1 > 1$, $\varepsilon_1 > 0$ such that for all $g_0 \in H^{(2,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{(2,0)} \leq \varepsilon_1,$$

the Cauchy problem admits a global solution satisfying

$$\|g\|_{L^\infty([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))} + \|\mathcal{H}^{\frac{1}{2}}(1-\mathbf{P})g\|_{L^2([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))} + \|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}(1-\mathbf{P})g\|_{L^2([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))} \leq c_1\|g_0\|_{(2,0)},$$

where $\mathbf{P}$ denotes the $L^2(\mathbb{R}^3_v)$ orthonormal projection on the space of collisional invariants.

Global existence

Proposition 7. Let $r=2$. Then, there exist some positive constants $c_1>1$, $\varepsilon_1>0$ such that for all $g_0\in H^{(2,0)}(\mathbb{R}^6_{x,v})$ satisfying

$$\|g_0\|_{(2,0)}\leq \varepsilon_1,$$

the Cauchy problem admits a global solution satisfying

$$\|g\|_{L^\infty([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))}+\|\mathcal{H}^{\frac{1}{2}}(1-\mathbf{P})g\|_{L^2([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))}$$
$$+\|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}(1-\mathbf{P})g\|_{L^2([0,+\infty[,H^{(2,0)}(\mathbb{R}^6_{x,v}))}\leq c_1\|g_0\|_{(2,0)},$$

where $\mathbf{P}$ denotes the $L^2(\mathbb{R}^3_v)$ orthonormal projection on the space of collisional invariants.

This can be extended to $r>3/2$.

6. Concluding Remarks

We can write Landau Equation (and also some other kinetic equations) as

6. Concluding Remarks

We can write Landau Equation (and also some other kinetic equations) as

$$\underbrace{\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial x}}_{\text{transport}} + \underbrace{\Lambda u}_{\text{diffusion}} = T(u, u).$$

6. Concluding Remarks

We can write Landau Equation (and also some other kinetic equations) as

$$\underbrace{\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial x}}_{\text{transport}} + \underbrace{\Lambda u}_{\text{diffusion}} = T(u,u).$$

• The matrix S is selfadjoint, the matrix Λ is selfadjoint non-negative, but **not positive-definite.**

6. Concluding Remarks

We can write Landau Equation (and also some other kinetic equations) as

$$\underbrace{\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial x}}_{\text{transport}} + \underbrace{\Lambda u}_{\text{diffusion}} = T(u,u).$$

• The matrix S is selfadjoint, the matrix Λ is selfadjoint non-negative, but **not positive-definite.**
• The commutator $[S, \Lambda]$ is not zero and the iterated commutators of S with Λ produce estimates for the components of u in the kernel of Λ. That non-commutation property is fundamental to the hypoellipticity properties of this equation.

6. Concluding Remarks

We can write Landau Equation (and also some other kinetic equations) as

$$\underbrace{\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial x}}_{\text{transport}} + \underbrace{\Lambda u}_{\text{diffusion}} = T(u, u).$$

• The matrix S is selfadjoint, the matrix Λ is selfadjoint non-negative, but **not positive-definite.**

• The commutator $[S, \Lambda]$ is not zero and the iterated commutators of S with Λ produce estimates for the components of u in the kernel of Λ. That non-commutation property is fundamental to the hypoellipticity properties of this equation.

• The matrix Λ is diagonal in a suitably chosen Hilbertian basis and the expression of the **quadratic term** $T(u, u)$ **is explicit** in that basis. The calculations are indeed intricate, but the special functions involved are rather classical.

6. Concluding Remarks

We can write Landau Equation (and also some other kinetic equations) as

$$\underbrace{\frac{\partial u}{\partial t} + S\frac{\partial u}{\partial x}}_{\text{transport}} + \underbrace{\Lambda u}_{\text{diffusion}} = T(u,u).$$

• The matrix S is selfadjoint, the matrix Λ is selfadjoint non-negative, but **not positive-definite.**
• The commutator $[S, \Lambda]$ is not zero and the iterated commutators of S with Λ produce estimates for the components of u in the kernel of Λ. That non-commutation property is fundamental to the hypoellipticity properties of this equation.
• The matrix Λ is diagonal in a suitably chosen Hilbertian basis and the expression of the **quadratic term** $T(u, u)$ **is explicit** in that basis. The calculations are indeed intricate, but the special functions involved are rather classical.

It would be interesting to check some numerical simulations using the above expression, maybe starting with the simpler Kac equation.

A slight variation would be to consider that the density f is singular when it vanishes. Setting $f = e^{-\phi}$, we get

$$\partial_t \phi + v \cdot \partial_x \phi + e^{\phi} Q\big(e^{-\phi}, e^{-\phi}\big) = 0.$$

An upside of that version is that non-negativity of the density is guaranteed, even in a perturbation mode with $\phi = \phi_0 + \varepsilon\psi$, yielding for $e^{-\phi_0} = \mu$,

$$\partial_t \psi + v \cdot \partial_x \psi + \mu^{-1} e^{\varepsilon\psi} \varepsilon^{-1} Q\big(\mu e^{-\varepsilon\psi}, \mu e^{-\varepsilon\psi}\big) = 0.$$

The H theorem has a simpler expression and proof in that framework with

$$\frac{d}{dt} \iint \phi e^{-\phi} dxdv \geq 0,$$

since with $S = \iint \phi e^{-\phi} dxdv$ we have

$$\begin{aligned}\dot{S} = \iint (\dot{\phi} - \phi\dot{\phi}) e^{-\phi} dxdv &= \iint e^{-\phi} (\phi - 1)\big(v \cdot \partial_x \phi + e^{\phi} Q\big(e^{-\phi}, e^{-\phi}\big)\big) dxdv \\ &= \iint (\phi - 1) Q\big(e^{-\phi}, e^{-\phi}\big) dxdv = \iint \phi Q\big(e^{-\phi}, e^{-\phi}\big) dxdv \geq 0.\end{aligned}$$

Thank you for your attention

Hermite functions

The standard one-dimensional Hermite functions are given by,

$$\phi_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \Big(x - \frac{d}{dx} \Big)^n (e^{-\frac{x^2}{2}}), \quad x \in \mathbb{R}, \; n \geq 0,$$

and define an orthonormal basis of $L^2(\mathbb{R})$. Setting for $n \geq 0$, $\alpha = (\alpha_j)_{1 \leq j \leq 3} \in \mathbb{N}^3$, $x \in \mathbb{R}$, $v = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$\psi_n(x) = 2^{-1/4} \phi_n(2^{-1/2} x), \quad \psi_n = \frac{1}{\sqrt{n!}} \Big(\frac{x}{2} - \frac{d}{dx} \Big)^n \psi_0, \quad \Psi_\alpha(v) = \prod_{j=1}^{3} \psi_{\alpha_j}(v_j), \tag{13}$$

we get that the family $(\Psi_\alpha)_{\alpha \in \mathbb{N}^3}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by the eigenfunctions of the Harmonic Oscillator

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4} = \sum_{k \geq 0} \Big(k + \frac{3}{2} \Big) \mathbb{P}_k, \quad \mathrm{Id} = \sum_{k \geq 0} \mathbb{P}_k, \tag{14}$$

where $\mathbb{P}_k$ denotes the orthogonal projection onto $\mathcal{E}_k = \mathrm{Span}\{\Psi_\alpha\}_{\alpha \in \mathbb{N}^3, |\alpha| = k}$.

Laplace-Beltrami operator on $\mathbb{S}^2$

Using spherical coordinates

$$\begin{cases} x = r\cos\beta\sin\alpha, \\ y = r\sin\beta\sin\alpha, \\ z = r\cos\alpha, \end{cases} \quad \begin{cases} r = \sqrt{x^2+y^2+z^2}, \\ \alpha = \operatorname{Im}\operatorname{Log}(z+i\sqrt{x^2+y^2}) \in (0,\pi) \quad \text{is the zenithal angle,} \\ \beta = \operatorname{Im}\operatorname{Log}(x+iy) \in (-\pi,\pi) \quad \text{is the azimuth,} \end{cases}$$

we get that

$$r^2\Delta_{\mathbb{R}^3} = (r\partial_r)^2 + (r\partial_r) + \Delta_{\mathbb{S}^2}, \quad \Delta_{\mathbb{S}^2} = \partial_\alpha^2 + \frac{1}{\sin^2\alpha}\partial_\beta^2 + \frac{1}{\tan\alpha}\partial_\alpha.$$

The space of spherical harmonics $\mathcal{Y}_l$ with degree l in three dimensions is defined as the space of homogeneous harmonic polynomials with degree l in three variables, $\dim\mathcal{Y}_l = 2l+1$,

$$-\Delta_{\mathbb{S}^2} Y = l(l+1)Y, \quad \text{for } Y \in \mathcal{Y}_l.$$

The real spherical harmonics $Y_l^m(\sigma)$ with $l \in \mathbb{N}$, $-l \leq m \leq l$, are defined as $Y_0^0(\sigma) = (4\pi)^{-1/2}$ and for any $l \geq 1$,

$$Y_l^m(\sigma) = \begin{cases} \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\alpha), & \text{if } m = 0 \\ \left(\frac{2l+1}{2\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2} P_l^m(\cos\alpha)\cos m\beta & \text{if } m = 1, ..., l \\ \left(\frac{2l+1}{2\pi}\frac{(l+m)!}{(l-m)!}\right)^{1/2} P_l^{-m}(\cos\alpha)\sin m\beta & \text{if } m = -l, ..., -1, \end{cases}$$

where P_l stands for the l-th Legendre polynomial and P_l^m the associated Legendre functions of the first kind of order l and degree m. We recall that

$$P_l(t) = \frac{(-1)^l}{2^l l!}\left(\frac{d}{dt}\right)^l (1-t^2)^l,$$

$$P_l^m(t) = (-1)^m(1-t^2)^{m/2}\left(\frac{d}{dt}\right)^m P_l(t).$$

Laguerre Polynomials

The generalized Laguerre polynomials are given by

$$L_n^{\gamma}(t) = \frac{t^{-\gamma} e^t}{n!} \left(\frac{d}{dt} \right)^n \left(e^{-t} t^{n+\gamma} \right).$$

We set for any $n, l \geq 0$, $-l \leq m \leq l$,

$$\varphi_{n,l,m}(v) = 2^{-3/4} \Big(\frac{2n!}{\Gamma(n+l+\frac{3}{2})} \Big)^{1/2} \Big(\frac{|v|}{\sqrt{2}} \Big)^l L_n^{l+\frac{1}{2}} \Big(\frac{|v|^2}{2} \Big) e^{-\frac{|v|^2}{4}} Y_l^m \Big(\frac{v}{|v|} \Big),$$

The family $(\varphi_{n,l,m})_{n,l \geq 0, |m| \leq l}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$ composed by eigenvectors of the harmonic oscillator and the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^2$,

$$\Big(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2} \Big) \varphi_{n,l,m} = (2n+l) \varphi_{n,l,m}, \quad -\Delta_{\mathbb{S}^2} \varphi_{n,l,m} = l(l+1) \varphi_{n,l,m}.$$