Short time regularization of diffusive inhomogeneous kinetic equations

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on recent works with R. Alexandre, W.-X. Li, L. Thomann, D. Tonon and I. Tristani

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Introduction

We look at a system described by a density of particles $0 \leq f(t,x,v)$ with $t \geq 0$, $x \in \mathbb{T}^3$ or $\mathbb{R}^3$ and $v \in \mathbb{R}^3$.

Inhomogeneous kinetic equations :

$$\partial_t f + v.\nabla_x f = C(f), \qquad f|_{t=0} = f^0$$

This problem has a long history (Maxwell, Boltzmann, Laudau).

Focus on models when the collision kernel has some diffusion properties

Possible models of diffusive collision kernels $C(f)$ may be

–> Bilinear : Q_B Boltzmann without cutoff, Q_L Landau

–> Linear : L_K Kolmogorov, L_{FP} Fokker-Planck, L_B linearized Boltzmann, L_L linearized Landau

For example the historical Kolmogorov equation reads

$$\partial_t f + v.\nabla_x f = \Delta_v f,$$

–> *hypoellipticity* : Solutions are known to be smooth for positive time

Natural questions :

- Is it true for others models ?
- What are the applications ?
- Are there quantitative estimates ?

Hypoellipticity

Consider the Kolmogorov equation

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x + \Delta_v.$$

The theory of (type II) hypoelliptic operators by Hörmander (1967) says that if $U \subset \mathbb{R}^6_{x,v}$ open bounded and $u \in \mathcal{C}^\infty_0(U)$ then

subelliptic estimate

$$\|u\|_s^2 \leq C(\|\Lambda u\|_0^2 + \|u\|_0^2) \quad \text{with } s = 2/3$$

Optimal because only $k = 1$ commutator is needed :

$$-\Lambda = X_0 + \sum X_j^* X_j \quad \text{and} \quad \left(X_0, X_j, Y_j \stackrel{\text{def}}{=} [X_j, X_0]\right)$$

span the whole tangent space $T\mathbb{R}^{2n}$ and $s = 2/(2k+1)$.

General remarks about the preceding result :

- Many methods to get this result (mention Kohn where $s = 1/4$, Hörmander, Helffer-Nourrigat, Rotschild-Stein,....).
- In general local methods.
- $-\Lambda$ not selfadjoint, nor elliptic.

From kinetic considerations we would like :

- Explicit methods and constants.
- Robust methods (apply to other models).
- Look at the heat $t \longrightarrow S_\Lambda(t) f_0$
- Measuring precisely the gain of regularity for the Cauchy problem

First results on the example of the Fokker-Planck equation

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x + \nabla_v.(\nabla_v + v) \qquad L_{FP} = \nabla_v.(\nabla_v + v)$$

In three steps :

- Global maximal explicit subelliptic estimate (H.-Nier 02, Helffer-Nier 05) in suitable exponentially weighted spaces :

$$\left\| |D_v|^2 f \right\|^2 + \left\| |D_x|^{2/3} f \right\|^2 + \|f\|^2$$
$$\lesssim \|\Lambda f\|^2 + \|f\|^2 \lesssim \left\| |D_v|^2 f \right\|^2 + \left\| |v|^2 f \right\|^2 + \left\| |D_x|^2 f \right\|^2 + \|f\|^2$$

- Deduce that the spectrum of $-\Lambda \geq 0$ is in $\left\{ |\mathrm{Im}\,(z)| \lesssim (\mathrm{Re}\,(z))^3 \right\}$ and get a resolvent estimate outside : cuspidal operators
- Use a Cauchy integral formula

$$S_\Lambda(t) f_0 = \frac{1}{2i\pi} \int_\Gamma e^{-tz} (z + \Lambda)^{-1} f_0 \, dz$$

Using this method

Theorem

$$\textit{for all } r \in \mathbb{R}, \quad \|S_\Lambda(t) f_0\|_{H^{r,r}_{x,v}} \leq \frac{C_r}{t^{N_r}} \|f_0\|_{H^{-r,-r}_{x,v}}$$

- Done for FP in $\mathbb{R}^3$ (H.-Nier 02), chains of oscillators (step 2, Eckmann-Hairer 03) general quadratic models (Hitrik-Pravda Starov-Viola 15)...
- Robust proof
- Sometimes sufficient for applications
- But not optimal, decay should depend on directions :
  1. Melher Formulas (Green kernels)
  2. Old result concerning Subunit balls, harmonic analysis (Fefferman 83, Coulhon-Saloff Coste-Varopoulos 92)
  3. Next section.

First examples and Lyapunov functionals

The basic heat equation example

$$\partial_t f - \Delta_v f = 0, \qquad \Lambda = \Delta_v$$

for a density $f(t, v)$ (forget variable x for a moment). Consider a time-dependant functional

$$\mathcal{H}(t, g) = \|g\|^2 + 2t\,\|\nabla_v g\|^2$$

$$\frac{d}{dt}\mathcal{H}(t, f(t)) = -2\,\|\nabla_v f(t)\|^2 + 2\,\|\nabla_v f(t)\|^2 - 2t\,\|\Delta_v f(t)\|^2 \leq 0$$

So that $\|\nabla_v f(t)\|^2 \leq \frac{C_1}{t}\,\|f_0\|^2$ which yields for $\Lambda = \Delta_v$

$$\|S_\Lambda(t) f_0\|_{H^1_v} \leq \frac{C_2}{t^{1/2}}\,\|f_0\|_{L^2_v}$$

We shall do the same for inhomogeneous models using the commutation identity $[\nabla_v, v.\nabla_x] = \nabla_x$.

Consider now the full (conjugated) Fokker Planck equation

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x - (-\nabla_v + v).\nabla_v \qquad L_{FP} = -(-\nabla_v + v).\nabla_v$$

For $C > D > E > 1$ to be defined later on, we define the functional

$$\mathcal{H}(t, g) = C \|g\|^2 + Dt \|\nabla_v g\|^2 + Et^2 \langle \nabla_v g, \nabla_x g \rangle + t^3 \|\nabla_x g\|^2 .$$

(where the norms are in $L^2(d\mu)$, μ is the Gaussian in velocity).

Then for C, D, E well chosen, we check similarly that

$$\frac{d}{dt}\mathcal{H}(t, f(t)) \leq 0.$$

First note that if $E^2 < D$, the crossed term is controlled by the two others. We have just modified a (time-dependant) norm in H^1.

Some Computations in a simpler case.

▷ First term

$$\frac{d}{dt}\|f\|^2 = 2\langle \partial_t f, f\rangle = -2\langle v\nabla_x f, f\rangle - 2\langle(-\nabla_v + v)\nabla_v f, f\rangle = -2\|\nabla_v f\|^2$$

▷ Second term

$$\begin{aligned}\frac{d}{dt}\|\nabla_v f\|^2 &= 2\langle \nabla_v(\partial_t f), \nabla_v f\rangle \\ &= -2\langle \nabla_v(v\nabla_x f + (-\nabla_v + v)\nabla_v f), \nabla_v f\rangle \\ &= -2\langle v\nabla_x\nabla_v f, \nabla_v f\rangle - 2\langle [\nabla_v, v\nabla_x] f, \nabla_v f\rangle - 2\langle \nabla_v(-\nabla_v + v)\nabla_v f \\ &= -2\langle \nabla_x f, \nabla_v f\rangle - 2\|(-\nabla_v + v)\nabla_v f\|^2\end{aligned}$$

▷ Last term

$$\frac{d}{dt}\|\nabla_x f\|^2 = -2\|\nabla_v\nabla_x f\|^2$$

▷ Third important term

$$\frac{d}{dt}\langle \nabla_x f, \nabla_v f\rangle$$
$$= -\langle \nabla_x(v\nabla_x f + (-\nabla_v + v)\nabla_v f), \nabla_v f\rangle - \langle \nabla_x f, \nabla_v(v\nabla_x f + (-\nabla_v + v)\nabla_v f)\rangle$$
$$= -\langle v\nabla_x(\nabla_x f), \nabla_v f\rangle - \langle (-\nabla_v + v)\nabla_v f, \nabla_x\nabla_v f\rangle$$
$$- \langle \nabla_x f, [\nabla_v, v\nabla_x]\, f\rangle - \langle \nabla_x f, v\nabla_x\nabla_v f\rangle$$
$$- \langle \nabla_x f, [\nabla_v, (-\nabla_v + v)]\, \nabla_v f\rangle - \langle (-\nabla_v + v)\nabla_v f, \nabla_x\nabla_v f\rangle .$$

we have

$$\langle v\nabla_x\nabla_x f, \nabla_v f\rangle + \langle \nabla_x f, v\nabla_x\nabla_v f\rangle = 0.$$

and

$$[\nabla_v, (-\nabla_v + v)] = 1$$

so that

$$\frac{d}{dt}\langle \nabla_x f, \nabla_v f\rangle = -\|\nabla_x f\|^2 + 2\langle (-\nabla_v + v)\nabla_v f, \nabla_x\nabla_v f\rangle - \langle \nabla_x f, \nabla_v f\rangle .$$

▷ Entropy dissipation inequality (simplest case)

$$\frac{d}{dt}\mathcal{H}(1,f(t)) = -2C\|\nabla_v f\|^2 - 2D\|(-\nabla_v + v)\nabla_v f\|^2 - E\|\nabla_x f\|^2 - 2\|\nabla_x \nabla_v f$$
$$-2(D+E)\langle \nabla_x f, \nabla_v f\rangle - 2E\langle (-\nabla_v + v)\nabla_v f, \nabla_x \nabla_v f\rangle .$$

Therefore, using Cauchy-Schwartz : for $1 < E < D < C$ well chosen,

$$\frac{d}{dt}\mathcal{H}(1,f(t)) \leq 0$$

The same occurs with t instead of 1 inside the definition of $\mathcal{H}$. This method, developed first in (H. 05)) gives for any $t \in [0,1)$

Theorem

$$\|S_\Lambda(t)h_0\|_{L^2_x H^1_v} \leq \frac{C}{t^{1/2}}\|h_0\|_{L^2_{x,v}}, \qquad \|S_\Lambda(t)h_0\|_{H^1_x L^2_v} \leq \frac{C_1}{t^{3/2}}\|h_0\|_{L^2_{x,v}}.$$

The Fractional Kolmogorov case reads

$$\partial_t f = \Lambda f \qquad \text{with} \qquad \Lambda = -v.\nabla_x - (1-\Delta_v)^{s/2} \qquad L_{FK} = -(1-\Delta_v)^{s/2}$$

The same procedure can be applied and we get

Theorem (H-Tonon-Tristani '17)

$$\|S_\Lambda(t)h_0\|_{L^2_x H^s_v} \leq \frac{C}{t^{1/2}}\|h_0\|_{L^2_{x,v}}, \qquad \|S_\Lambda(t)h_0\|_{H^s_x L^2_v} \leq \frac{C_1}{t^{(1+2s)/2}}\|h_0\|_{L^2_{x,v}}.$$

The Boltzmann without cutoff case

The Boltzmann equation in the torus reads

$$\nabla_t f + v \cdot \nabla_x f = Q_B(f, f)$$

$$\underbrace{(v', v'_*)}_{\text{before collision}} \rightleftarrows \underbrace{(v, v_*)}_{\text{after collision}}$$

- Conservation of momentum and energy :

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

- Parametrization of (v', v'_*) by an element $\sigma \in S^2$.

$$Q_B(g, f)(v) = \int_{\mathbb{R}^3 \times \mathrm{Spect}^2} \underbrace{B(v - v_*, \sigma)}_{\text{collision kernel}} \left(\underbrace{f(v')\, g(v'_*)}_{\text{"appearing"}} - \underbrace{f(v)\, g(v_*)}_{\text{"disappearing"}} \right) dv_*\, d\sigma$$

- Particles interacting according to a repulsive potential of the form $\phi(r) = r^{-(p-1)}$, $p \in (2, +\infty)$. We only deal with the case $p > 5$ (hard potentials).
- The collision kernel $B(v - v_*, \sigma)$ satisfies

$$B(v - v_*, \sigma) = C|v - v_*|^\gamma \, b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma$$

- b is not integrable on S^2 :

$$\sin\theta \, b(\cos\theta) \approx \theta^{-1-2s}, \quad s = \frac{1}{p-1}, \quad \forall \theta \in (0, \pi/2].$$

  For hard potentials $s \in (0, 1/4)$.
- The kinetic factor $|v - v_*|^\gamma$ satisfies $\gamma = \frac{p-5}{p-1}$. For hard potentials $\gamma > 0$.

Near the equilibrium $f = \mu + h$, the Linearized Boltzmann equation reads

$$\partial_t h = \underbrace{-v \cdot \nabla_x h + Q(\mu, h) + Q(h, \mu)}_{\Lambda h = \text{linear part}} \quad (+ \underbrace{Q(h,h)}_{\text{Nonlinear part}}).$$

Theorem (H-Tonon-Tristani '17)

We have for k large enough and $k' > k$ large enough :

$$\|S_\Lambda(t)h_0\|_{L^2_x H^s_v(\langle v\rangle^k)} \leq \frac{C_s}{t^{1/2}} \|h_0\|_{L^2_{x,v}(\langle v\rangle^{k'})}, \quad \forall\, t \in (0,1],$$

and

$$\|S_\Lambda(t)h_0\|_{H^s_x L^2_v(\langle v\rangle^k)} \leq \frac{C_r}{t^{(1+2s)/2}} \|h_0\|_{L^2_{x,v}(\langle v\rangle^{k'})}, \quad \forall\, t \in (0,1].$$

$\hookrightarrow$ Key point to develop a perturbative Cauchy theory.

$\hookrightarrow$ tools In the spirit of [Alexandre-H-Li 15] for the Boltzmann case.

Elements of proof :

- apart from a regularizing part, the linearized Boltzmann Kernel looks like (with $D_v = i^{-1}\nabla_v$)

$$\Lambda \sim -v.\nabla_x + \langle v\rangle^{\gamma}\left(1 + |D_v|^2 + |D_v \wedge v|^2 + |v|^2\right)^s$$

- we can use microlocal/pseudo-differential techniques to estimate the collision part. Anyway, due to bad symbolic properties, Weyl has to be replaced by Wick quantization (tools developed by Lerner) and Garding inequality by unconditional positivity.
- from Alexandre-H-Li 15, we use symbolic estimates and built a close-to-semiclassical class of symbols.
- a Lyapunov functional very similar to the one of the fractional FP can be built.

The Vlasov-Poisson-Fokker-Planck equation reads

$$\begin{cases} \partial_t f + v.\nabla_x f - (\varepsilon_0 E + \nabla_x V).\nabla_v f - \gamma \nabla_v.\left(\nabla_v + v\right) f = 0, \\ E(t,x) = -\dfrac{1}{|\mathbb{S}^{d-1}|}\dfrac{x}{|x|^d} \star_x \rho(t,x), \quad \text{where} \quad \rho(t,x) = \displaystyle\int f(t,x,v)dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

We can write $-\Lambda = v.\nabla_x f - \nabla_x V.\nabla_v f - \gamma \nabla_v.\left(\nabla_v + v\right) f$ and consider the Duhamel formula

$$f(t) = S_\Lambda(t) f_0 + \varepsilon_0 \int_0^t E \underbrace{S_\Lambda(t-s)\nabla_v}_{\text{integrable singularity}} f(s)\, ds.$$

By fixed point Theorem, this yields a result of existence and trend to the equilibrium in $H^{a,a}$ spaces with $a \in (1/2, 2/3)$ (H.-Thomann 15)

This type of regularizing result can also be crucial in the Cauchy theory in large spaces as recently proposed by Gualdani-Mischler-Mouhot 15. We consider here the Boltzmann without cutoff case :

Considering the Boltzmann model, we have

- Conservation of mass, momentum and energy :

$$\int_{\mathbb{R}^3} Q(f,f)(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dv = 0$$

- Entropy inequality (H-theorem) :

$$D(f) := -\int_{\mathbb{R}^3} Q(f,f)(v) \log f(v)\, dv \geq 0$$

and

$$D(f) = 0 \Leftrightarrow f \text{ is a Gaussian in } v)$$

A priori estimates

We fix $\mu = (2\pi)^{-3/2} e^{-|v|^2/2}$.

In what follows, we shall consider initial data f_0 with same mass, momentum, energy as μ

A priori estimates : if f_t is solution of the Boltzmann equation associated to f_0 with finite mass, energy and entropy then :

$$\sup_{t \geq 0} \int (1 + |v|^2 + |\log f_t|) \, f_t \, dx \, dv + \int_0^\infty D(f_s) \, ds < \infty.$$

and

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_t \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dx \, dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dx \, dv.$$

Does $f_t \xrightarrow[t \to \infty]{} \mu$? If yes, what is the rate of convergence ? is it explicit ?

Main results

$$\partial_t f + v \cdot \nabla_x f = Q_B(f,f) \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3.$$

Theorem (H.-Tonon-Tristani '17)

If f_0 is close enough to the equilibrium μ, then there exists a global solution $f \in L^\infty_t(X)$ to the Boltzmann equation. Moreover, for any $0 < \lambda < \lambda_\star$ there exists $C > 0$ such that

$$\forall\, t \geq 0, \quad \|f_t - \mu\|_X \leq C\, e^{-\lambda t}\, \|f_0 - \mu\|_X.$$

- X is a Sobolev space of type $H^3_x L^2_v(\langle v \rangle^k)$ with k large enough.
- $\lambda_\star > 0$ is the optimal rate given by the semigroup decay of the associated linearized operator.
- Key element of the proof in the enlargment theory : Duhamel formula for $\Lambda = A + B$

$$S_\Lambda(t) = S_B(t) + \int_0^t S_\Lambda(t-s) A S_B(s)\, ds.$$

- ⋆ Global renormalized solutions with a defect measure : DiPerna-Lions 89, Villani 96, Alexandre-Villani 04
- ⋆ Perturbative solutions in $H^{\ell}_{x,v}(\mu^{-1/2})$
  - Landau equation : Guo 02, Mouhot-Neumann 06
  - Boltzmann equation : Gressman-Strain 11, Alexandre et al. 11
- ⋆ Solutions in Sobolev spaces with polynomial weight for the Boltzmann equation : He-Jiang 17, Alonso-Gamba-Taskovic 17
- ⋆ Improvements :
  - The weights are less restrictive.
  - Less assumptions on the derivatives.

Thank you !