Linear Boltzmann Equation and Fractional Diffusion

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COMPENDIUM ON (FRACTIONAL) DIFFUSION LIMITS

FOR THE LINEAR BOLTZMANN EQUATION

Diffusion Approximation of Linear Boltzmann Equation

Linear Boltzmann equation, for a system of monokinetic particles, with isotropic scattering

$$(\partial_t + \omega \cdot \nabla_x + \sigma) f(t,x,\omega) = \sigma \langle f \rangle (t,x) := \sigma \int_{\mathbf{S}^{d-1}} f(t,x,\omega') \tfrac{d\omega'}{|\mathbf{S}^{d-1}|}$$

Distribution function $f \equiv f(t,x,\omega) \geq 0$, scattering rate $\sigma \gg 1$

$$\begin{aligned} f(t,x,v) =& \langle f \rangle (t,x) - \frac{1}{\sigma} \omega \cdot \nabla_x f(t,x,\omega) - \frac{1}{\sigma} \partial_t f(t,x,\omega) \\ \simeq& \langle f \rangle (t,x) - \frac{1}{\sigma} \omega \cdot \nabla_x \langle f \rangle (t,x) - \frac{1}{\sigma} \partial_t \langle f \rangle (t,x) + o(1/\sigma) \end{aligned}$$

Two Fundamental Ingredients

(a) Local conservation of particle number (exact)

$$\partial_t\langle f\rangle + \mathrm{div}_x\langle \omega f\rangle = 0$$

(b) Fourier law for the current (approximate)

$$\begin{aligned}\langle \omega f\rangle &\simeq \langle\omega\rangle\langle f\rangle(t,x) - \frac{1}{\sigma}\langle\omega\otimes\omega\rangle\cdot\nabla_x\langle f\rangle(t,x) - \frac{1}{\sigma}\langle\omega\rangle\partial_t\langle f\rangle(t,x)\\ &\simeq -\frac{1}{\sigma}\langle\omega\otimes\omega\rangle\cdot\nabla_x\langle f\rangle(t,x) = \frac{1}{d\sigma}\nabla_x\langle f\rangle(t,x)\end{aligned}$$

since

$$\langle\omega\rangle = 0\,,\qquad \langle\omega\otimes\omega\rangle = \tfrac{1}{d}\langle|\omega|^2\rangle I = \tfrac{1}{d}I$$

Conclusion $f \simeq \langle f\rangle \simeq \rho(t,x)$, with

$$\partial_t\rho(t,x) = \frac{1}{d\sigma}\Delta_x\rho(t,x)$$

Boundary Condition

•If the linear Boltzmann equation is posed on a C^1 domain Ω with outward unit normal n_x, the Dirichlet condition for incoming particles

$$f(t,x,\omega) = R_b(x)\,, \quad x \in \partial\Omega\,, \quad \omega \cdot n_x < 0$$

leads to the Dirichlet condition for the diffusion equation at order 0

$$\rho(t,x) = R_b(x)\,, \quad x \in \partial\Omega$$

•At order 1 in $1/\sigma$, one has a Robin boundary condition

$$\rho(t,x) + \frac{L}{\sigma}\partial_n \rho(t,x) = R_b(x)\,, \quad x \in \partial\Omega$$

Extrapolation coefficient is computed in terms of a boundary layer equation (Milne problem); $L := .7104$ if $d = 3$

Linear Boltzmann equation of the form

$$(\partial_t + v \cdot \nabla_x + \sigma) f(t,x,v) = M(v) \int_{\mathbf{R}^d} f(t,x,w)dw\,, \quad x,v \in \mathbf{R}^d$$

Probability density on $\mathbf{R}^d$

$$M \equiv M(v) = M(-v) \sim |v|^{-d-2\alpha}\,, \quad \alpha \in (0,1)$$

In the limit as $\sigma \gg 1$

$$f \sim \rho(t/\sigma^{2\alpha}, x) M(v)\,, \quad \partial_t \rho = c(-\Delta_x)^\alpha \rho$$

Standard diffusion regime **impossible** since the would-be diffusion coefficient

$$\int |v|^2 M(v) dv = \infty$$

Non LTE radiative transfer

$$(\sigma(\nu) - h(\epsilon)\mu\partial_x)I(x,\mu,\nu) = (1-\epsilon)\sigma(\nu)\langle\!\langle\sigma(\nu)I\rangle\!\rangle + \epsilon\sigma(\nu)Q(x)$$

where

$$\langle\!\langle f\rangle\!\rangle := \tfrac{1}{2}\int_0^\infty\int_{-1}^1 f(\mu,\nu)d\mu d\nu$$

In the case

$$\sigma(\nu) := \tfrac{2}{\sqrt{\pi}}e^{-\nu^2}, \qquad \text{set } h(\epsilon) := \epsilon\sqrt{|\ln\epsilon|}$$

the source function

$$S(x) := (1-\epsilon)\langle\!\langle\sigma(\nu)I\rangle\!\rangle(x) + \epsilon Q(x)$$

satisfies

$$S(x) + \sqrt{-\Delta}S(x) = Q(x)$$

Two Mechanisms Driving to Fractional Diffusion

(1) In the work of Mellet (see also similar ideas in the work of several authors — Aceves-Sanchez, Cesbron, Mischler, Mouhot, Schmeiser... — the scattering rate is constant, but the fractional diffusion comes from the "heavy tail" in the equilibrium distribution, so that the current is not proportional to the gradient of the density (Fick's law does not hold)

(2) In the work of H. and U. Frisch, the scattering rate is not uniformly large in ν — thus the background medium is opaque for low ν and "transparent" for high ν; fractional diffusion sets in as a compromise between standard diffusion and ballistic transport

A NEW ROUTE TO FRACTIONAL DIFFUSION LIMITS FOR THE LINEAR BOLTZMANN EQUATION

Caffarelli-Silvestre Approach to the Fractional Laplacian

Let $f \in L^p(\mathbf{T}^d)$, with $1 \le p < \infty$, and let $0 < \gamma < 2$. Then

$$(-\Delta_x)^{\gamma/2} f(x) = -\partial_y F(x,0)$$

with $F \equiv F(x,y) \in C_b([0,\infty); L^p(\mathbf{T}^d))$ the **harmonic extension**

$$\begin{cases} \Delta_x F(x,y) + \gamma^2 c_\gamma^{\gamma/2} y^{2-2/\gamma} \partial_y^2 F(x,y) = 0\,, & y > 0 \\ F(x,0) = f(x)\,, & x \in \mathbf{T}^d \end{cases}$$

with

$$c_\gamma = 2^{-\gamma} |\Gamma(-\gamma/2)| / \Gamma(\gamma/2)$$

Remarks (a) If $\gamma = 1$, then $c_\gamma = 1$
(b) In this case, fractional Laplacian=Dirichlet-to-Neuman operator
Not always true...

Position variable $z=(x,y)$ with $x\in\mathbf{R}^2$ and $y>0$
Gray radiative intensity in direction $\omega\in\mathbf{S}^2$ denoted $f\equiv f(z,\omega)\geq 0$

$$\omega\cdot\nabla_z f(z,\omega)=\sigma\int_{\mathbf{S}^2}p(\omega,\omega')(f(z,\omega')-f(z,\omega))d\omega'\,,\quad y>0$$

Thomson type scattering, at (frequency independent) rate $\sigma>0$
Scattering transition probability $p\equiv p(\omega,\omega')\in L^2(\mathbf{S}^2\times\mathbf{S}^2)$ s.t.

$$p(\omega,\omega')=p(\omega',\omega)>0\,,\qquad\int_{\mathbf{S}^2}p(\omega,\omega')d\omega'=1$$
$$p(Q\omega,Q\omega')=p(\omega,\omega')\,,\qquad Q\in O_3(\mathbf{R})\,,\ \omega,\omega'\in\mathbf{S}^2$$

Examples
(a) $p(\omega,\omega')=\frac{1}{4\pi}$ (isotropic scattering)
(b) $p(\omega,\omega')=\frac{3}{16\pi}(1+(\omega\cdot\omega')^2)$ (Rayleigh phase function)

Boundary Condition

Incoming radiation at $y = 0$: Lambert reflection+ isotropic source — e.g. black body

$$f(x,0,\omega) = R(x) + \frac{\alpha}{\pi}\int_{\omega'_y<0} f(x,0,\omega')|\omega'_y|d\omega' \,, \qquad \omega_y > 0$$

Example

Source $R(x) = aT(x)^4$ where $T(x)$ =temperature at x (black-body)

$$a := \frac{8\pi^5 k_B^4}{15c^2\hbar^3} \quad \text{(Boltzmann-Stefan constant)}$$

$\alpha :=$ albedo coefficient

Scaling Assumptions

(a) High scattering regime $\sigma \gg 1$

(b) High albedo regime $1 - \alpha \sim 1/\kappa\sigma$

Realized by taking

$$\alpha = \frac{\kappa\sigma}{1+\kappa\sigma}\,, \qquad S(x) = (1+\kappa\sigma)R(x)$$

Notation

$$\langle\phi\rangle := \tfrac{1}{4\pi}\int_{\mathbf{S}^2}\phi(\omega)d\omega\,, \qquad \langle\!\langle\psi\rangle\!\rangle_{\pm} := 4\langle\omega_y^{\pm}\psi\rangle$$

with

$$\omega_y^{\pm} := \max(\pm\omega_y, 0)$$

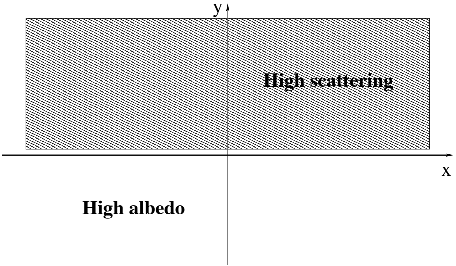

Figure: The interface problem: diffusion approximation holds in the high scattering region; Lambert's reflection law holds on the interface with the high albedo lower half-space. The radiation pressure on the interface is governed by a fractional diffusion equation involving $\sqrt{-\Delta}$

Rescaled equation

$$(BVP)\quad \begin{cases} \omega\cdot\nabla_z f(z,\omega)+\sigma f(z,\omega)=\sigma\int_{\mathbf{S}^2}p(\omega,\omega')f(z,\omega')d\omega' \\ \\ f(x,0,\omega)=\dfrac{S(x)}{1+\kappa\sigma}+\dfrac{\kappa\sigma}{1+\kappa\sigma}\langle\!\langle f\rangle\!\rangle_-(x,0) \end{cases}$$

Equivalent form of BC

$$f(x,0,\omega)=S(x)-4\kappa\sigma\langle\omega f\rangle(x,0)\,,\qquad \omega_y>0$$

since

$$(1+\kappa\sigma)f(x,0,\omega)=f(x,0,\omega)+4\kappa\sigma\langle\omega^+ f\rangle(x,0)\,,\qquad \omega_y>0$$

Main Result

Denote

$$Z=\mathbf{T}^2\times(0,+\infty)\,,\quad \mathcal{L}f(z,\omega):=f(z,\omega)-\int_{\mathbf{S}^2}p(\omega,\omega')f(z,\omega')d\omega'$$

Thm A
Let $S\in W^{1,\infty}(\mathbf{T}^2)$. For each $\sigma>0$ there exists a unique solution to the BVP satisfying

$$f_\sigma\in L^\infty(Z\times\mathbf{S}^2)$$

s.t. for all $j=1,2$

$$\|f_\sigma\|_{L^\infty(Z\times\mathbf{S}^2)}\le\|S\|_{L^\infty(\mathbf{T}^2)}$$
$$\|\partial_{x_j}f_\sigma\|_{L^\infty(Z\times\mathbf{S}^2)}\le\|\partial_{x_j}S\|_{L^\infty(\mathbf{T}^2)}$$

Lemma There exists a unique $\Omega \in L^2(\mathbf{S}^2)$ s.t.

$$(\mathcal{L}\Omega)(\omega) = \omega\,, \quad \langle \Omega \rangle = 0$$

Example Assuming that $p(\omega, \omega') = q((\omega \cdot \omega')^2)$, one finds $\Omega(\omega) = \omega$

Thm B
In the limit as $\sigma \to \infty$, one has

$$f_\sigma\big|_{y=0} \rightharpoonup R \equiv R(x) \quad \text{in } L^2(\mathbf{T}^2 \times \mathbf{S}^2; |\omega_y| dx d\omega)$$

where $R \equiv R(x)$ is the solution to the fractional diffusion equation

$$R(x) + \tfrac{4}{3}\kappa\langle \omega \cdot \Omega \rangle(-\Delta_x)^{1/2} R(x) = S(x)\,, \qquad x \in \mathbf{T}^2$$

Strategy

(1) To prove that

$$f_\sigma \rightharpoonup \rho \text{ in } L^\infty(Z \times \mathbf{S}^2) \quad \text{and} \quad \sigma(f - \langle f \rangle) \rightharpoonup -\Omega \cdot \nabla_z \rho \text{ in } L^2(Z \times \mathbf{S}^2)$$

where

$$\rho \equiv \rho(z) \in L^\infty(Z)\,, \qquad \text{and} \quad \nabla\rho \in L^2(Z)$$

(2) ... and that

$$\begin{cases} \Delta\rho(z) = 0\,, & z \in Z \\ \rho(x,0) - \frac{4}{3}\kappa\langle \Omega \cdot \omega \rangle \partial_y \rho(x,0) = S(x)\,, & x \in \mathbf{T}^2 \end{cases}$$

(3) Finally, to prove that

$$f_\sigma\big|_{y=0} \rightharpoonup R \equiv R(x) \text{ in } L^2(\mathbf{T}^2, \omega_y^- d\omega dx)$$
$$\sigma\langle \omega_y f \rangle \rightharpoonup -\tfrac{1}{3}\langle \Omega \cdot \omega \rangle \partial_y \rho\big|_{y=0} \text{ in } H^{-1/2}(\mathbf{T}^2)$$

KEY POINTS/IDEAS IN THE PROOF

Lemma The operator $\mathcal{L}$ is bounded, self-adjoint, Fredholm on $L^2(\mathbf{S}^2)$:

$$\ker \mathcal{L} = \mathbf{R}\,, \quad \operatorname{ran} \mathcal{L} = \mathbf{R}^\perp\,, \quad (\phi|\mathcal{L}\phi)_{L^2} \ge \mu\|\phi - \langle\phi\rangle\|_{L^2}^2$$

Proof: the operator $I - \mathcal{L}$ is Hilbert-Schmidt because $p \in L^2((\mathbf{S}^2)^2)$; that the nullspace is the set of constants follows from $p(\omega,\omega') > 0$

Entropy flux

$$\begin{aligned}\sigma\mu\int_{\mathbf{T}^2}\|f_\sigma - \langle f_\sigma\rangle\|_{L^2_\omega}^2\,dx &\le \sigma\int_{\mathbf{T}^2}(f_\sigma|\mathcal{L}f_\sigma)_{L^2_\omega}\,dx\\ &= -\frac{d}{dy}\int_{\mathbf{T}^2}\langle\tfrac{1}{2}\omega_y f_\sigma^2\rangle(x,y)dx\end{aligned}$$

Integrating in y

$$\sigma\mu\int_Z\|f_\sigma - \langle f_\sigma\rangle\|_{L^2_\omega}^2\,dz \le \|f_\sigma\|_{L^\infty_{z\omega}}^2$$

Uniqueness

Assume that $S \equiv 0$.

(1) by H Theorem

$$\int_{\mathbf{T}^2} \langle \tfrac{1}{2}\omega_y f_\sigma^2 \rangle (x,y) dx \text{ is nonincreasing}$$

(2) splitting $f_\sigma = f_\sigma - \langle f_\sigma \rangle + \langle f_\sigma \rangle$ and expanding square norm

$$\int_{\mathbf{T}^2} \langle \tfrac{1}{2}\omega_y f_\sigma^2 \rangle (x,y) dx \in L^1_y + L^2_y \Rightarrow \quad \to 0 \text{ as } y \to \infty$$

(3) since $S \equiv 0$

$$\sigma\mu \int_Z \| f_\sigma - \langle f_\sigma \rangle \|^2_{L^2_\omega} dz = 0$$

(4) hence

$$0 = f_\sigma - \langle f_\sigma \rangle = \sigma \mathcal{L} f_\sigma = \omega \cdot \nabla_z f_\sigma + BC \Rightarrow f_\sigma = 0$$

Bound on $f_\sigma - \langle f_\sigma \rangle$

(1) Straightforward application of H Theorem gives

$$\|f_\sigma - \langle f_\sigma \rangle\|^2_{L^2_{z,\omega}} \leq \frac{\|S\|_{L^\infty_x}}{\mu\sigma}$$

Gives $O(1/\sqrt{\sigma})$, but $O(1/\sigma)$ is needed

(2) Write BVP for $g_\sigma(z,\omega) = f_\sigma(z,\omega) - F_\sigma(z)$, where

$$F_\sigma(z) := \left(\tfrac{S(x)}{1+\kappa\sigma} + \tfrac{\kappa\sigma}{1+\kappa\sigma} \langle\!\langle f_\sigma \rangle\!\rangle_-(x,0) \right)(1-y)^2_+$$

One gets

$$\omega \cdot \nabla_z g_\sigma + \sigma \mathcal{L} g_\sigma = -\omega \cdot \nabla F_\sigma(z)\,, \qquad g_\sigma\big|_{y=0,\omega_y>0} = 0$$

H Thm applied to the BVP satisfied by g_σ implies that

$$\|\sigma(f_\sigma - \langle f_\sigma \rangle)\|_{L^2_{z,\omega}} = \|\sigma(g_\sigma - \langle g_\sigma \rangle)\|_{L^2_{z,\omega}} \leq C\|S\|^2_{W^{1,\infty}(\mathbf{T}^2)}$$

Write weak formulation of BVP for the test function

$$\sigma\phi(z) + \Omega(\omega)\cdot\nabla_z\phi(z)\,, \quad \phi\in C_c^\infty(\overline{Z})$$

Passing to the limit as $\sigma\to\infty$, one gets

$$\tfrac{3}{4}\int(S(x)-\rho(x,0))\phi(x,0)dx = \kappa\langle\Omega\cdot\omega\rangle\int_Z\nabla\phi\cdot\nabla\rho(z)dz$$

One recognizes in this identity the variational formulation of

$$\begin{cases}\Delta\rho(z)=0\,, & z\in Z\\ \rho(x,0)-\tfrac{4}{3}\kappa\langle\Omega\cdot\omega\rangle\partial_y\rho(x,0)=S(x)\,, & x\in\mathbf{T}^2\end{cases}$$

Finally one obtains the fractional diffusion equation by the harmonic extension representation of $\sqrt{-\Delta}$

FINAL REMARKS/EXTENSIONS

More General Domains

If Z is a smooth (bounded) domain of $\mathbf{R}^{d+1}$ with boundary ∂Z and outward normal field denoted n_z for $z \in \partial Z$, the diffusion limit of

$$(BVP') \qquad \begin{cases} \omega \cdot \nabla_z f_\sigma + \sigma \mathcal{L} f_\sigma = 0 & \text{in } Z \\ f_\sigma\big|_{\Gamma_-} = \frac{S(x)}{1+\kappa\sigma} + \frac{\kappa\sigma|\mathbf{S}^d|}{(1+\kappa\sigma)|\mathbf{B}^d|}\langle(\omega \cdot n_z)_+ f_\sigma\rangle\big|_{\Gamma} \end{cases}$$

is

$$\begin{cases} \Delta\rho(z) = 0\,, & z \in Z \\ \rho\big|_{\partial Z} + \frac{|\mathbf{B}^{d+1}|}{|\mathbf{B}^d|}\kappa\langle\Omega \cdot \omega\rangle \partial_n \rho\big|_{\partial Z} = S \end{cases}$$

Denoting by Λ the Dirichlet-to-Neuman operator on ∂Z, we conclude that $R := \rho\big|_{\partial Z}$ satisfies the equation

$$R + \frac{|\mathbf{B}^{d+1}|}{|\mathbf{B}^d|}\kappa\langle\Omega \cdot \omega\rangle \Lambda R = S$$

When is $\Lambda = \text{Const.}\sqrt{-\Delta}$?

Dirichlet-to-Neumann operator Λ: $H^{1/2}(\partial Z) \to H^{-1/2}(\partial Z)$ given by

$$\Lambda u = \partial_n U\big|_{\partial Z} \quad \text{where } \begin{cases} \Delta U = 0 & \text{in } Z \\ U\big|_{\partial Z} = u \end{cases}$$

so that

$$\langle \Lambda u, v \rangle_{H^{-1/2}, H^{1/2}} = \int_Z \nabla U(z) \cdot \nabla v(z) dz$$

Examples

(a) If $Z = \mathbf{R}^d \times (0, \infty)$ (half-space), then $\Lambda = \sqrt{-\Delta^{\mathbf{R}^d}}$

(b) If $Z = \mathbf{B}^2$ (unit disk in $\mathbf{R}^2$), then $\Lambda = \sqrt{-\Delta^{\mathbf{S}^1}}$

(c) If $Z = \mathbf{B}^d$ (unit ball in $\mathbf{R}^d$), then

$$\Lambda = -\tfrac{d-1}{2} + \sqrt{\tfrac{(d-1)^2}{4} - \Delta^{\mathbf{S}^d}} \le \sqrt{-\Delta^{\mathbf{S}^d}}$$

Open Questions

(a) Is there an analogue of this result for the linearized Boltzmann equation of the kinetic theory of gases? Need to get rid of maximum principle in the argument...

(b) Can one get powers of $-\Delta$ other than $\sqrt{-\Delta}$? Requires **strong** anisotropy in the diffusion limit for the linear Boltzmann equation

Linearizing about $M \equiv M_{(1,0,1)}$ the Boltzmann equation in the half-space $x_3 > 0$ with accomodation BC at $x_3 = 0$:

$$\begin{cases} v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} \mathcal{B}(F, F) \\ F(x, v) = (1 - \alpha) F(x, v^R) + \alpha \sqrt{\frac{2\pi}{T_w}} M_{(1,0,T_w)} \int_{\bar{v}_3 < 0} F(x, \bar{v}) |\bar{v}_3| d\bar{v} \end{cases}$$

with $F = M(1 + g)$ and $T_w = 1 + \theta_w$ (assuming $g, \theta_w \ll 1$) gives

$$\begin{cases} v \cdot \nabla_x g + \frac{1}{\mathrm{Kn}} \mathcal{L} g = 0 \\ g(x, v) = (1 - \alpha) g(x, v^R) + \alpha \theta_w \frac{1}{2}(|v|^2 - 4) \\ \qquad + \alpha \sqrt{2\pi} \int_{\bar{v}_3 < 0} g(x, \bar{v}) |\bar{v}_3| M d\bar{v} \end{cases}$$

Asymptotic Analysis (Aoki-Inamuro-Onishi JPSJ 1979)

Scaling $\alpha = \mathrm{Kn} = \epsilon \ll 1$... unlike $1 - \alpha \ll 1$ in Radiative Transfer!

Hilbert expansion + Knudsen layer

$$g_\epsilon(x, v) = g_0(x, v) + \epsilon \left(g_1(x, v) + \phi(x_1, x_2, \tfrac{x_3}{\epsilon}, v) \right) + \dots$$

One finds

$$\begin{aligned} g_0(x, v) =& u_0(x) \cdot v + \theta_0(x) \tfrac{1}{2}(|v|^2 - 5) \\ g_1(x, v) =& \rho_1(x) + u_1(x) \cdot v + \theta_1(x) \tfrac{1}{2}(|v|^2 - 3) \\ & - \nabla_x u_0(x) : \hat{A}(v) - \nabla_x \theta_0(x) \cdot \hat{B}(v) \end{aligned}$$

where

$$\mathcal{L}\hat{A} = v^{\otimes 2} - \tfrac{1}{3}|v|^2 I \,, \quad \mathcal{L}\hat{B} = v \tfrac{1}{2}(|v|^2 - 5)$$

... while ϕ solves the Milne problem

$$\begin{cases} v_3\partial_z\phi + \mathcal{L}\phi = 0 \text{ for } x_3 > 0 \\ \phi(0,v) - \phi(0,v^R) = \Phi(v) \end{cases} \qquad \text{and} \quad \lim_{z\to+\infty} \phi = 0$$

where

$$\Phi := (g_1^R - g_1 - g_0^R)\big|_{x_3=0} + \sqrt{2\pi}\int_{\bar v_3<0} g_0\big|_{x_3=0}|\bar v_3| M d\bar v + \theta_w \frac{|v|^2-4}{2}$$

Vanishing condition at infinity

$$\int \Phi(v) v_3 \chi(v) M dv = 0 \quad \text{for } \chi(v) \equiv 1, v_1, v_2, |v|^2$$

[Aoki-Inamuro-Onishi JPSJ1979, FG-Perthame-Sulem ARMA1988]

One finds that

$$\begin{cases} \nabla p = \nu \Delta u_0 \,, \qquad \operatorname{div} u_0 = 0 \,, & x_3 > 0 \\ (u_{0,j} - \sqrt{2\pi}\nu \partial_{x_3} u_{0,j})\big|_{x_3=0} = u_{0,3}\big|_{x_3=0} = 0, & j = 1, 2 \end{cases}$$

so that $u_0 = 0$, and

$$\begin{cases} \kappa \Delta \theta_0 = 0 \,, & x_3 > 0 \\ (\theta_0 - \frac{\sqrt{2\pi}}{2} \kappa \partial_{x_3} \theta_0)\big|_{x_3=0} = \theta_w \end{cases}$$

Setting $\Theta = \theta_0\big|_{x_3=0}$, this is equivalent by harmonic extension to

$$\Theta(x_1, x_2) + \frac{\sqrt{2\pi}}{2} \kappa \sqrt{-\Delta}\Theta(x_1, x_2) = \theta_w(x_1, x_2)$$