Global in time and mixing solutions for the Incompressible Porous Media equation (IPM).

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Incompressible porous media equation in $\mathbb{R}^2$

Two-dimensional mass balance equation in porous media (2D IPM)

$$\begin{cases} \boxed{\begin{array}{l} \rho_t + u \cdot \nabla \rho = 0 \\ \frac{\mu}{\kappa} u = -\nabla p - (0, g\rho) \\ \operatorname{div} u = 0 \end{array}} \end{cases}$$

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Remark: let $\mu = \kappa = g = 1$

- $u(x) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} (-2\frac{y_1 y_2}{|y|^4}, \frac{y_1^2 - y_2^2}{|y|^4})\, \rho(x-y) dy \,-\, \frac{1}{2}\,(0, \rho(x))\,,$
- $||\rho||_{L^p}(t) = ||\rho||_{L^p}(0) \quad p \in [1, \infty] \Longrightarrow ||u||_{L^p}(t) \leq C \quad p \in (1, \infty)$
- $(\partial_t + u \cdot \nabla)\nabla^{\perp}\rho = (\nabla u)\nabla^{\perp}\rho.$

Two Settings: Global existence and Mixing solutions

- Muskat: The density ρ takes takes two different constant values

$$\rho(x,t) = \begin{cases} \rho^1 & x \in \Omega^1(t) \\ \rho^2 & x \in \Omega^2(t) \end{cases}$$

- Smooth solutions $\rho(x,t) \in H^k$ for $k \geq 3$.

New Results

- Muskat equation Global existence with arbitrarily large slope.

  with O. Lazar. arXiv 2018

- Weak Solutions Mixing solutions.

  with A. Castro and D. Faraco. recent updated version of arXiv:1605.04822

- Smooth initial data Global existence of quasi-stratified solutions for the confined IPM equation

  with A. Castro and D. Lear. arXiv 2018

Muskat: Contour equation

We consider

$$\rho(x,t) = \begin{cases} \rho^1 & x \in \Omega^1(t) \\ \rho^2 & x \in \Omega^2(t) \end{cases}$$

with

$$\partial\Omega^j(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \,:\, \alpha \in \mathbb{R}\}.$$

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We consider:

1. Open curves vanishing at infinity

$$\lim_{\alpha\to\infty} (z(\alpha,t) - (\alpha,0)) = 0,$$

2. Periodic curves in the space variable

$$z(\alpha + 2k\pi, t) = z(\alpha,t) + 2k\pi(1,0).$$

3. Closed curves $\Rightarrow$ Unstable regime.

Muskat: Contour equation

Darcy's law:

$$u = -\nabla p - (0, \rho) \Rightarrow \nabla^{\perp} \cdot u = -\partial_{x_1} \rho.$$

$$\nabla^{\perp} \cdot u(x,t) = -(\rho^2 - \rho^1)\partial_\alpha z_2(\alpha,t)\delta(x - z(\alpha,t)).$$

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Biot-Savart:

$$u(x,t) = -\frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \partial_\alpha z_2(\beta,t) d\beta,$$

for $x \neq z(\alpha,t)$.

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$$\|u\|_{L^2}(t) < \infty.$$

Muskat: Contour equation

It yields

$$z_t(\alpha) = \frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{z_1(\alpha) - z_1(\beta)}{|z(\alpha) - z(\beta)|^2} (\partial_\alpha z(\alpha) - \partial_\beta z(\beta)) d\beta.$$

- SOLUTIONS OF THE MUSKAT PROBLEM $\Longrightarrow$ WEAK SOLUTIONS OF IPM

Contour equation as a graph

- The equation for a graph $z(\alpha, t) = (\alpha, f(\alpha, t))$.

$$\alpha_t = \frac{\rho^2 - \rho^1}{2\pi} \int_{\mathbb{R}} \frac{(\alpha - \beta)(\partial_\alpha \alpha - \partial_\beta \beta)}{(\alpha - \beta)^2 + (f(\alpha) - f(\beta))^2} d\beta$$

$$(0 = \quad 0)$$

$$\boxed{f_t(\alpha) = \frac{\rho^2 - \rho^1}{2\pi} \int_{\mathbb{R}} \frac{(\alpha - \beta)(\partial_\alpha f(\alpha) - \partial_\beta f(\beta))}{(\alpha - \beta)^2 + (f(\alpha) - f(\beta))^2} d\beta}$$

with initial data

$$z_1(\alpha, 0) = \alpha$$
$$z_2(\alpha, 0) = f(\alpha, 0) = f_0(\alpha).$$

The linearized equation

$$f^L_t(\alpha,t) = -\frac{\rho^2-\rho^1}{2}\Lambda(f^L)(\alpha,t), \qquad \Lambda = (-\Delta)^{1/2}.$$

Fourier transform:

$$\widehat{f^L}(\xi,t) = \widehat{f_0}(\xi,t)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi|t\Big).$$

- $\rho^2 > \rho^1$ stable case,
- $\rho^2 < \rho^1$ unstable case.

Local existence theory

For a general interface

$$\partial\Omega^j(t) = \{\mathbf{z}(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)), \quad \alpha \in \mathbb{R}\}$$

after taking k derivatives $(k \geq 3)$ it can be shown that

$$\partial_t \partial_\alpha^k \mathbf{z}(\alpha,t) = -\underbrace{(\rho^2 - \rho^1)\frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha \mathbf{z}(\alpha,t)|^2}}_{\sigma(\alpha,t)\equiv R-T} \Lambda \partial_\alpha^k \mathbf{z}(\alpha,t) + \text{l.o.t.}$$

Thus we can distinguish three regimes:

- Stable regime: $\sigma > 0 \implies$ the denser fluid is always below.
  The Muskat problem is locally well-posed in time in Sobolev's spaces.
- Fully unstable regime: $\sigma < 0 \implies$ the denser fluid is always above.
  The Muskat problem is ill-posed in Sobolev's spaces.
- Partial unstable regime: σ has not a defined sign $\implies$ there is a part of the interface where the denser fluid is above.

Local existence results in the stable regime

- D.C. and F. Gancedo (2007). Local existence in H^3 (and ill-posedness for $\rho^2 < \rho^1$).
- A. Cheng, R. Granero and S. Shkoller (2016). Local existence in H^2.
- P. Constantin, F. Gancedo, R. Shvydkoy and V. Vicol (2017). Local existence in $W^{2,p}$ for p>1.
- B-V. Matioc (arxiv). Local existence in $H^{\frac{3}{2}+\epsilon}$.

Conserved quantities in the stable regime: $(z_1, z_2) = (\alpha, f(\alpha, t))$

- $\int f(\alpha, t)\, d\alpha = \int f_0(\alpha)\, d\alpha.$

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- Maximum principle for the L^2-norm

$$||f(\cdot, t)||^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)^2\right) d\alpha d\beta = ||f_0||^2_{L^2(\mathbb{R})}$$

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Compare with the linear case

$$||f(\cdot, t)||_{L^2(\mathbb{R})} + \int_0^T \underbrace{\int_{\mathbb{R}} \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)^2 d\alpha d\beta}_{=\int_{\mathbb{R}} f(x)\Lambda f(x) dx = ||\Lambda^{\frac{1}{2}} f(\cdot,t)||^2_{L^2(\mathbb{R})}} dt = ||f_0||^2_{L^2(\mathbb{R})}$$

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But

$$\frac{1}{2}\int_{\mathbb{R}} \int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)^2\right) d\alpha d\beta \leq C||f(\cdot, t)||_{L^1}$$

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- Maximum principle: $||f||_{L^\infty}(t) \leq ||f||_{L^\infty}(0)$.

  Periodic case:

$$||f - \frac{1}{2\pi}\int_{\mathbb{T}} f_0 d\alpha||_{L^\infty}(t) \leq ||f_0 - \frac{1}{2\pi}\int_{\mathbb{T}} f_0 d\alpha||_{L^\infty} e^{-Ct}.$$

  Flat at infinity: $||f||_{L^\infty}(t) \leq \dfrac{||f_0||_{L^\infty}}{1+Ct}$.

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  Flat at infinity: $\|f\|_{L^\infty}(t) \leq \dfrac{\|f_0\|_{L^\infty}}{1 + Ct}$.

- Maximum principle: If $\|f_\alpha\|_{L^\infty}(0) < 1$ then $\|f_\alpha\|_{L^\infty}(t) \leq \|f_\alpha\|_{L^\infty}(0)$.

Global existence for $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} < 1$

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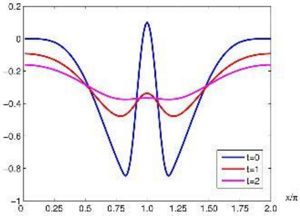
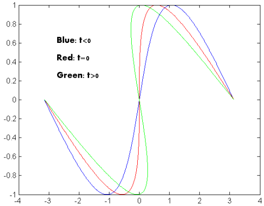
- S. Cameron (arxiv2017). Global classical solutions if $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} < 1$.

What happens if $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} > 1$ (with finite energy)?

- Numerical simulations of Turning (i.e. shift of stability) by Maria López-Fernández





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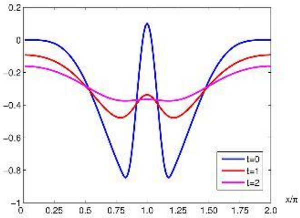

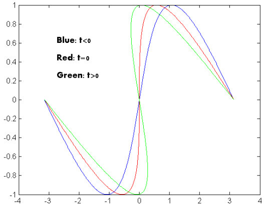

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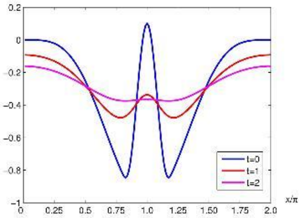

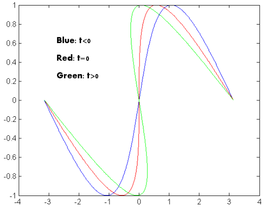

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- Numerical evidence of turning with $||\partial_\alpha f_0||_{L^\infty} = 22$ by J. Gómez-Serrano. Is there a turning for $||\partial_\alpha f_0||_{L^\infty} = 1 + \epsilon$?

What happens after Turning?

- In the stable regime a solution of Muskat becomes immediately real-analytic and then passes to the unstable regime in finite time. Moreover, the Cauchy-Kowalewski theorem shows that a real-analytic Muskat solution continues to exist for a short time after the turnover.

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- Breakdown of smoothness (2013): There exist interfaces of the Muskat problem such that after turnover their smoothness breaks down (is not C^4). Joint work with A. Castro, C. Fefferman and F. Gancedo.
- Double shift of stability (2017): Turning stable-unstable-stable (also unstable-stable-unstable). Joint work with J. Gómez-Serrano and A. Zlatos.

Global existence for arbitrarily large slope

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Theorem

Assume $f_0 \in H^{5/2}$ with $\|f_0\|_{\dot{H}^{3/2}}$ small enough, then, there exists a unique strong solution f which verifies $f \in L^\infty([0,T], H^{3/2}) \cap L^2([0,T], \dot{H}^{5/2})$, for all $T > 0$.

Joint work with O. Lazar.

Global existence for arbitrarily large slope: proof

Main steps of the proof:

- The proof is based on the use of a new formulation of the Muskat equation that involves oscillatory terms as well as a careful use of Besov space techniques.
- $$f_t(t,x) = \frac{\rho}{\pi} P.V. \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f)\, d\delta\, d\alpha$$
  $$f(0,x) = f_0(x).$$

  where $\Delta_\alpha f \equiv \frac{f(x,t) - f(x-\alpha,t)}{\alpha}$.

Global existence for arbitrarily large slope: proof

- A priori estimates in $\dot{H}^{3/2}$:

$$\begin{aligned}\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} &= \int \mathcal{H}f_{xx}\int \partial_{xx}\Delta_\alpha f\int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx\\ &- \int \mathcal{H}f_{xx}\int (\partial_x\Delta_\alpha f)^2\int_0^\infty \delta e^{-\delta}\sin(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx\\ &= I_1 + I_2\end{aligned}$$

We can estimate

$$|I_2| \leq \|f\|^2_{\dot{H}^2}\|f\|_{\dot{H}^{3/2}}$$

and the most singular term is I_1

$$|I_1| \lesssim \|f\|^2_{\dot{H}^2}(\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}) - \pi\|f\|^2_{\dot{H}^2} + \pi\frac{K^2}{1+K^2}\|f\|^2_{\dot{H}^2}$$

where $K = \|f_x\|_{L^\infty L^\infty}$.

Global existence for arbitrarily large slope: proof

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$$
\begin{aligned}
\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} &= \int \mathcal{H}f_{xx} \int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\, d\delta\, d\alpha\, dx \\
&- \int \mathcal{H}f_{xx} \int (\partial_x\Delta_\alpha f)^2 \int_0^\infty \delta e^{-\delta}\sin(\delta\Delta_\alpha f(x))\, d\delta\, d\alpha\, dx \\
&= I_1 + I_2
\end{aligned}
$$

  We can estimate

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  and the most singular term is I_1

$$|I_1| \lesssim \|f\|^2_{\dot{H}^2}(\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}) - \pi\|f\|^2_{\dot{H}^2} + \pi\frac{K^2}{1+K^2}\|f\|^2_{\dot{H}^2}$$

  where $K = \|f_x\|_{L^\infty L^\infty}$.

- Then

$$\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} + \frac{\pi}{1+K^2}\|f\|^2_{\dot{H}^2} \leq C\|f\|^2_{\dot{H}^2}\left(\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}\right)$$

Global existence for arbitrarily large slope: proof

- Similar a priori estimates in $\dot{H}^{5/2}$:

Lemma

Let $T > 0$ and $f_0 \in \dot{H}^{5/2} \cap \dot{H}^{3/2}$ so that $\|f_0\|_{\dot{H}^{3/2}} < C(\|f_{0,x}\|_{L^\infty})$, then we have

$$
\begin{aligned}
\|f\|^2_{\dot{H}^{5/2}}(T) \quad + \quad & \frac{\pi}{1+M^2}\int_0^T \|f\|^2_{\dot{H}^3}\, ds \\
\lesssim \quad & \|f_0\|_{\dot{H}^{5/2}} + \left(\|f\|_{L^\infty([0,T],\dot{H}^{3/2})} + \|f\|^2_{L^\infty([0,T],\dot{H}^{3/2})}\right)\int_0^T \|f\|^2_{\dot{H}^3}\, ds
\end{aligned}
$$

where M is the space-time Lipschitz norm of f.

MIXING SOLUTIONS

Fully unstable regime $\rho_1 > \rho_2$

Previous work:

- Ill-posedness in Sobolev spaces (D.C.-F. Gancedo 2007)

Theorem

Let $s > 3/2$*, then for any* $\varepsilon > 0$ *there exists a solution* f *of the Muskat equation with* $\rho_1 > \rho_2$ *and* $0 < \delta < \varepsilon$ *such that* $\|f\|_{H^s}(0) \leq \varepsilon$ *and* $\|f\|_{H^s}(\delta) = \infty$.

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- CAN WE STILL FIND WEAK SOLUTIONS FOR IPM?

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Let $s > 3/2$, then for any $\varepsilon > 0$ there exists a solution f of the Muskat equation with $\rho_1 > \rho_2$ and $0 < \delta < \varepsilon$ such that $\|f\|_{H^s}(0) \leq \varepsilon$ and $\|f\|_{H^s}(\delta) = \infty$.

- CAN WE STILL FIND WEAK SOLUTIONS FOR IPM?

- Mixing solutions from a flat interface (Székelyhidi 2012)

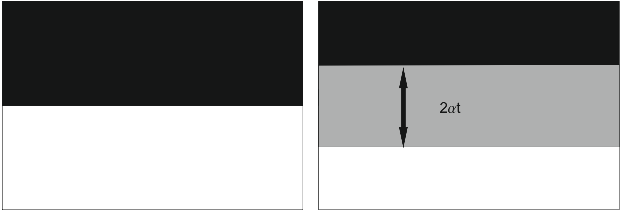
$$\rho(\mathbf{x}, t) = \begin{cases} +1 & \mathbf{x} \in \{x_2 \geq \alpha t\} \\ \pm 1 & \mathbf{x} \in \{-\alpha t < x_2 < \alpha t\} \\ -1 & \mathbf{x} \in \{x_2 < -\alpha t\} \end{cases}$$

for $\alpha \in (0, 2)$.

Székelyhidi's construction

Remarks:

- The solution starts in the fully unstable regime being flat. There exist a solution for the Muskat equations: the flat interface is a stationary solutions
- There is a mixing zone: $\{-\alpha t < x_2 < \alpha t\}$.
- The solutions are not unique:
  - For different values of α (the speed of the opening of the mixing zone) we have different solutions.
  - For a fixed value of α, inside of the mixing zone there are infinitely many different densities.



The definition of a mixing solution

The density $\rho(\mathbf{x}, t)$ and the velocity $(\mathbf{x}, t)$ are a "mixing solution" of the IPM system if they are a weak solution and also there exist, for every $t \in [0, T]$, open simply connected domains $\Omega^{\pm}(t)$ and $\Omega_{mix}(t)$ with $\overline{\Omega^{+}} \cup \overline{\Omega^{-}} \cup \Omega_{mix} = \mathbb{R}^2$ such that, for almost every $(x, t) \in \mathbb{R}^2 \times [0, T]$, the following holds:

$$\rho(x,t) = \begin{cases} \rho^{\pm} & \text{in } \Omega^{\pm}(t) \\ (\rho - \rho^{+})(\rho - \rho^{-}) = 0 & \text{in } \Omega_{mix}(t) \end{cases} .$$

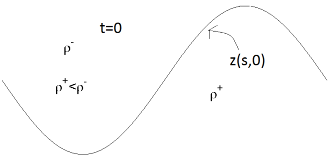
For every $r > 0, x \in \mathbb{R}^2, 0 < t < T\ B((x,t),\ r) \subset \cup_{0<t<T}\Omega_{mix}(t)$ it holds that

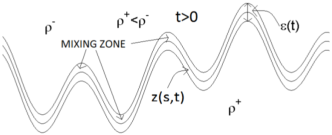
$$\int_B (\rho - \rho^{+}) \int_B (\rho - \rho^{-}) \neq 0$$

Main theorem: joint work with A. Castro and D. Faraco (arxiv2016)

Theorem

Let $\Gamma(0) = \{z^0(s) = (z_1^0(s), z_2^0(s)) \in \mathbb{R}^2\}$ *with* $z^0(s) - (s,0) \in H^5$*. We will assume that* $\Gamma(0)$ *is run from left to right and that* $\frac{\partial_s z_1^0(s)}{|\partial_s z^0(s)|} > 0$*. Let us suppose that* $\rho^+ < \rho^-$*. Then there exist infinitely many "mixing solutions" starting with the inital data of Muskat type given by* $\Gamma(0)$ *(in the fully unstable regime) for the IPM system.*





New results:

- Piecewise constant subsolutions. C. Forster and L. Székelyhidi (arxiv2017)
- Linear degraded mixing solutions. A. Castro, D. Faraco and F. Mengual (arxiv2018)
- Updated version with a variable width of the mixing zone (2018)

Subsolution implies Mixing solutions

Definition

We will say that (ρ, u, m) *is a subsolution of the IPM system if there exist open simply connected domains* $\Omega^{\pm}(t)$ *and* $\Omega_M(t)$ *with* $\overline{\Omega^+} \cup \overline{\Omega^-} \cup \Omega_M = \Omega$ *and such that the following holds:*

- *The flow is incompressible*

$$\nabla \cdot u = 0 \quad in\ \Omega.$$

- *In* Ω (ρ, u, m) *satisfy the equations*

$$\begin{aligned} \partial_t \rho + \nabla \cdot m &= 0 \\ \nabla^{\perp} \cdot u &= -\partial_{x_1} \rho. \end{aligned}$$

- The density and m satisfies

$$\rho(x,t) = \pm 1 \quad m = \rho u \quad \text{in } \Omega^{\pm}$$

  and it is continuous in Ω.

- In Ω_M the functions (ρ, u, m) are in the "Mixing hull"

$$\left| m - \rho u + \frac{1}{2}\left(0, 1 - \rho^2\right) \right|^2 < \left(\frac{1}{2}\left(1 - \rho^2\right) \right)^2$$
$$\rho^2 < 1$$

Laszlo Székelyhidi 2012 proved

Theorem

Given a subsolution $(\overline{\rho}, \overline{u}, \overline{m})$ there exist infinitely many mixing solutions $(\rho, u) \in L^\infty \times L^\infty$ such that $(\rho, u) = (\overline{\rho}, \overline{u})$ in $\Omega \setminus \Omega_M$.

Example of a Mixing solution

The mixing zone

$$\Omega_M = \{x \in \Omega : |x_2| < \alpha t\}.$$

then

$$\rho(x,t) = \begin{cases} -1 & x_2 < -\alpha t, \\ \frac{x_2}{2\alpha t} & |x_2| < \alpha t \\ 1 & x_2 > \alpha t, \end{cases}, \qquad m = (0, -\alpha(1-\rho^2)) \qquad u = (0,0)$$

is a subsolution with $\alpha \in (0,2)$.

Convex integration for IPM

- This work is based on a variant of the method of convex integration introduced for Euler equations by C. de Lellis and L. Székelyhidi Jr. Lack of uniqueness and Onsager's conjecture (E. Wiedemann, C. Bardos, A. Choffrut, P. Isett, T. Buckmaster, V. Vicol...).
- D. C., D. Faraco and F. Gancedo. Lack of uniqueness for IPM.
- R. Shvydkoy. Non-uniqueness for active scalars with a divergence free velocity given by a Fourier multiplier operator with an even symbol.
- L. Székelyhidi. Lack of uniqueness for IPM (computation of the Λ-convex hull). Flat mixing solutions.
- P. Isett, V. Vicol. Global existence of weak solutions for active scalars with multipliers that are not odd from arbitrarily smooth initial data. And there exist nontrivial solutions, compact support in time, having any Holder regularity $\rho \in C^{\alpha}_{t,x}$ with $\alpha < \frac{1}{9}$.

Constructing a subsolution (ρ, u, m) from $z_0 \in H^k$

We define the set $\Omega_M \subset \mathbb{R}^2$

$$\Omega_M = \{\mathbf{x} \in \mathbb{R}^2 \; : \; \mathbf{x} = \mathbf{x}(s,\lambda) \quad \text{for} \quad (s,\lambda) \in (-\infty,\infty) \times (-\varepsilon,\varepsilon)\}.$$

with

$$\mathbf{x}(s,\lambda) = z(s,t) + (0,\lambda)$$

We take

$$\rho(\mathbf{x}) = \begin{cases} \pm 1 & \text{in } \overline{\Omega^{\pm}} \\ \frac{\lambda}{\epsilon} & \text{in } \Omega_M \end{cases},$$

by Biot-Savart

$$u(\mathbf{x}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \partial_s z(s') \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{x_1 - x_1(s',\lambda')}{|\mathbf{x} - \mathbf{x}(s',\lambda')|^2} d\lambda' ds'.$$

Constructing a subsolution (ρ, u, m) from $z_0 \in H^k$

We take

$$m = \rho u - (\beta, \alpha)\left(1 - \rho^2\right)$$

then

$$\partial_t \rho + u \cdot \nabla \rho = \nabla \cdot \left((\beta, \alpha)\left(1 - \rho^2\right)\right)$$

Given $(\varepsilon(t), z(s,t))$ we have (ρ, u, m)

Constructing a subsolution (ρ, u, m) from $z_0 \in H^k$

If there is a solution $(\varepsilon(t), z(s,t))$ to the following system

$$\begin{aligned} \partial_t \mathbf{z}(s,t) &= \mathcal{M}[\mathbf{z},\varepsilon](s,t) & \qquad \mathbf{z}(s,0) &= \mathbf{z}^0(s) \\ \partial_t \varepsilon(t) &= c & \varepsilon(0) &= 0, \end{aligned}$$

where $c > 0$ is a constant and the velocity $\mathcal{M}[\mathbf{z},\varepsilon](s,t)$ is given by

$$\begin{aligned} &\mathcal{M}[\mathbf{z},\varepsilon](s,t) \\ &= -\frac{1}{4\varepsilon^2\pi}\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty}\int_{-\varepsilon}^{\varepsilon}\frac{(\partial_s \mathbf{z}(s) - \partial_s \mathbf{z}(s'))(z_1(s) - z_1(s'))}{|\mathbf{z}(s) - \mathbf{z}(s') + (\lambda - \lambda')(0,1)|^2} d\lambda' ds' d\lambda. \end{aligned}$$

then the solution is in the Mixing Hull.

Contour dynamics

- We have to solve the equation

$$\partial_t \mathbf{z}(s,t) = \mathcal{M}[\mathbf{z},\varepsilon](s,t) \qquad \mathbf{z}(s,0) = \mathbf{z}^0(s) \qquad (1)$$

$$\partial_t \varepsilon(t) = c \qquad \varepsilon(0) = 0, \qquad (2)$$

- We can use that the the interface can be parametrize as the graph a function $f(x,t)$,

$$(z_1(s,t), z_2(s,t)) = (x, f(x,t))$$

- Quasi-linearization. We take $\partial_x^5 f(x,t) = F(x,t)$. We can write

$$\partial_t F(x,t) = \int K_{\partial_x f(x,t)}(x-y)\partial_x F(y,t)dy + a(x,t)\partial_x F(y,t)dy + G(x,t) \quad (3)$$

where $G(x,t)$ and $a(x,t)$ are low order functions.

The kernel K_A

Taking $\varepsilon = t$ $(c = 1)$

$$\begin{aligned}
&K_A(y,t)\\
&= \frac{1}{4\pi t^2}\left\{-2Ay\arctan(A) + (2t+Ay)\arctan\left(\frac{2t+Ay}{y}\right)\right.\\
&+ (Ay-2t)\arctan\left(\frac{Ay-2t}{y}\right) + y\log\left(y^2(1+A^2)\right)\\
&\left.-\frac{y}{2}\log\left(y^2+(2t+Ay)^2\right) - \frac{y}{2}\log\left(y^2+(Ay-2t)^2\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&\hat{K}_A(\xi,t)\\
&= \frac{-i\operatorname{sign}(\xi)}{2\pi|\xi|t}\left\{1 + \frac{1}{4\pi|\xi|t}\left(e^{-4\pi|\xi|t\sigma}\left(\cos(4\pi|\xi|t\sigma A)\right.\right.\right.\\
&\qquad\left.\left.\left.-A\sin(4\pi|\xi|t\sigma A)\right) - 1\right)\right\}.
\end{aligned}$$

Existence of solutions $z(s,t) \in H^k$

Let $\mathbf{F}(s,t) = \partial_s^{5)} z(s,t)$ then

$$\partial_t \mathbf{F} = \mathcal{L} * \Lambda \mathbf{F} + a(s,t)\partial_s \mathbf{F} + l.o.t.$$

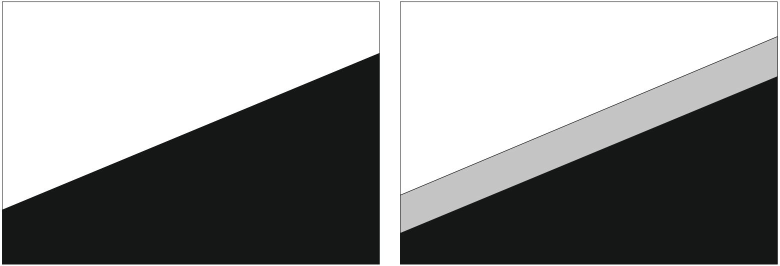
where $\widehat{\mathcal{L} * f}(\xi) \sim \frac{1}{1+t|\xi|}\hat{f}(\xi)$.

Goal: By choosing the correct energy we can prove

$$||F(t)||_{H^s} \leq C(T)||F^0||_{H^{s+1}}.$$

Mixing in the stable regime

We can obtain mixing in the stable regime.



SMOOTH SETTING

Goal:

Global Existence of small data with
finite energy and **bounded density**

IPM

$$\begin{cases} \partial_t \varrho + \mathbf{u} \cdot \nabla \varrho = 0 \\ \frac{\mu}{\kappa} \mathbf{u} = -\nabla P - g(0, \varrho) \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

On the Asymptotic Stability of Stationary Solutions of the Inviscid Incompressible Porous Medium Equation.
Tarek M. Elgindi. **ARMA**, 2017.

UNBOUNDED DENSITY

Stationary Solution:

$$\mathbf{u} = 0 \qquad \varrho \equiv \varrho(y)$$

The perturbation:

$$\varrho(x, y, t) = -y + \rho(x, y, t),$$
$$P(x, y, t) = \Pi(x, y, t) - \frac{1}{2} y^2.$$

Stratified Solution in a bounded domain

Our setting:

$$D = \mathbb{T} \times [-1, 1]$$

No-slip boundary condition:

$$\mathbf{u} \cdot \mathbf{n} = 0$$

Our perturbation:

$$\varrho(x, y, t) = -y + \rho(x, y, t),$$

$$P(x, y, t) = \Pi(x, y, t) - \frac{1}{2}y^2.$$

$$(\star) \begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = u_2 \\ \mathbf{u} = -\nabla \Pi - (0, \rho) \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

besides the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D.

Well-posedness

In order to solve our problem in the bounded domain Ω, in certain Sobolev space, we have to overcome new difficulties:

- To be able to handle the boundary terms that appear in the computations.

We can be bypass if our perturbation has a special structure.

$$X^k(\Omega) := \{f \in H^k(\Omega) : \partial_y^n f|_{\partial\Omega} = 0 \quad \text{for } n = 0, 2, 4, \dots\},$$
$$Y^k(\Omega) := \{f \in H^k(\Omega) : \partial_y^n f|_{\partial\Omega} = 0 \quad \text{for } n = 1, 3, 5, \dots\}$$

Local Existence

Theorem

Let $k \geq 3$ and $\rho(0) \in X^k(\Omega)$. There exists $T > 0$ and a unique solution $\rho \in C\left(0, T; X^k(\Omega)\right)$ of the perturbated IPM system $(\star)$ such that:

$$\sup_{0 \leq t < T} ||\rho||_{H^k(\Omega)}(t) \leq C\, ||\rho||_{H^k(\Omega)}(0).$$

Moreover, for all $t \in [0, T)$ we have:

$$||\rho||_{H^k}(t) \leq ||\rho||_{H^k}(0)\ \exp\left[\widetilde{C} \int_0^t \left(||\nabla\rho||_{L^\infty}(s) + ||\nabla \mathbf{u}||_{L^\infty}(s)\right)\, ds\right].$$

Key points of the proof:

- Galerkin approximation
- Properties of our ONB

Global existence: Energy Estimate

The following estimate holds for $k > 4$

$$\frac{1}{2}\partial_t ||\rho||^2_{H^k(D)} \lesssim ||\partial u_2||_{L^\infty(D)} \, ||\rho||^2_{H^k(D)} - \left(1 - ||\rho||_{H^k(D)}\right) \, ||\mathbf{u}||^2_{H^k(D)}$$
$$+ \text{BOUNDARY TERMS}$$

Solution Boundary Terms:

$$\rho(0) \in X^k \Longrightarrow \rho(t) \in X^k \quad \text{is preserved in time (local existence).}$$

$$\rho(t) \in X^k \Longrightarrow \begin{cases} u_2(t) \in X^k \\ u_1(t) \in Y^k \end{cases}$$

Global existence: Energy Estimate

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BOUNDARY TERMS$= 0$

The Boostraping

For $k > 4$:

$$\frac{1}{2}\partial_t||\rho||^2_{H^k(D)} \lesssim ||\partial u_2||_{L^\infty(D)}\, ||\rho||^2_{H^k(D)} - \left(1 - ||\rho||_{H^k(D)}\right)\, ||\mathbf{u}||^2_{H^k(D)}$$

Take: $||\rho||_{H^\kappa(D)}(0) \leq \varepsilon$

Bootstrap hypothesis: $||\rho||_{H^\kappa(D)}(t) \leq 4\varepsilon \qquad t \in [0, T]$

By Grönwall's inequality in $[0, T]$ we have:

$$||\rho||_{H^\kappa(D)}(t) \leq ||\rho||_{H^\kappa(D)}(0) \cdot \exp\left(\int_0^t ||\partial u_2||_{L^\infty(D)}(s)\, ds\right)$$

Bootstrap conclusion:

$$||\rho||_{H^\kappa(D)}(t) \leq 2\varepsilon \qquad t \in [0, T]$$

Linear Estimates

Linearized equation around $(\rho, \mathbf{u}) = (0, 0)$:

$$\begin{cases} \partial_t \rho &= u_2 \\ u_2 &= -\partial_y \Pi - \rho \end{cases} \qquad (x, y) \in D$$

Remark: If $D = \mathbb{R}^2$, $\mathbf{u} \equiv \mathbf{R}^\perp R_1 \rho$

$$\boxed{\partial_t \rho = R_1^2 \rho}$$

Linear Decay

$$||u_2||_{H^k(D)}(t) \lesssim \frac{||u_2||_{H^{k+\alpha}(D)}(0)}{(1+t)^{(\alpha+1)/4}}$$

Non-Linear Estimates

Perturbation of the linear system:

$$\partial_t u_2 - L u_2 = F \qquad F \equiv \mathbf{u} \cdot \nabla \rho$$
$$L u_2 \equiv -\partial_t \partial_y \Pi - u_2$$

Duhamel's formula:

$$u_2(t) = e^{L\,t} u_2(0) + \int_0^t e^{L\,(t-s)} \left(\mathbf{u} \cdot \nabla \rho\right)(s)\, ds$$

Non-linear Decay If $||\rho||_{H^{\kappa}(D)}(t) \leq 4\varepsilon$ for $t \in [0, T]$ with $\kappa \equiv \kappa(\alpha)$.

$$||\partial u_2||_{L^{\infty}(D)} \lesssim \frac{\varepsilon}{(1+t)^{(\alpha+1)/4}} \qquad t \in [0, T]$$

Finishing the proof

Hypothesis: $||\rho||_{H^\kappa(D)}(0) \leq \varepsilon$

$||\rho||_{H^\kappa(D)}(t) \leq 4\varepsilon$ for all $t \in [0, T]$

$$\begin{aligned} ||\rho||_{H^\kappa(D)}(t) &\leq ||\rho||_{H^\kappa(D)}(0) \cdot \exp\left(\int_0^t ||\partial u_2||_{L^\infty(D)}(s)\, ds\right) \\ &\leq \varepsilon \exp\left(\int_0^t \frac{C\varepsilon}{(1+s)^{(\alpha+1)/4}}\, ds\right) \\ &\leq \varepsilon \exp(\tilde{C}\varepsilon) \end{aligned}$$

Conclusion:

Take $0 < \varepsilon << 1 \Longrightarrow ||\rho||_{H^\kappa(D)}(t) \leq 2\varepsilon$ for all $t \in [0, T]$

A continuity Argument:

$$||\rho||_{H^\kappa(D)}(t) \leq 2\varepsilon \quad \text{for all } t \geq 0$$

THANK YOU