Singularities in Fluids

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- inviscid limit with vortex sheet data (with H. Nusenzveig-Lopes, M. Lopes, V. Vicol)

Hypersurfaces of Evolution

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The surface is assumed to be orientable, exterior normal is n. The vectors $\{n, f_{,1}, \cdots, f_{,d}\}$ computed at any $\alpha \in D$ form a basis of $\mathbb{R}^{d+1}$.

Velocity decomposition, evolution of normal

$$v = an + b^j f_{,j}$$

$$a = < n, v >, \qquad b^j = g^{jk} < f_{,k}, v >$$

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$$\frac{\partial}{\partial t} f_{,k} = a_{,k} n + a n_{,k} + b^j_{,k} f_{,j} + b^j f_{,jk},$$

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where h_{jk} are the coefficients of the second fundamental form II:

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Evolution of first and second fundamental forms

Recall $< f_{,pi}, f_{,j} >= [pi, j]$, the Christoffel symbols of the second kind $\Gamma^r_{pi} = g^{rj}[pi, j]$ and

$$b^r_{;i} = b^r_{,i} + \Gamma^r_{pi} b^p,$$

the covariant gradient of the tangent vector b. We obtain after calculations:

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$$\frac{\partial}{\partial t} II = \nabla\nabla a - a\, II(I^{-1})II + L_b(II)$$

where $\nabla\nabla a$ is the matrix:

$$a_{;kl} = a_{,kl} - \Gamma^p_{kl} a_{,p}.$$

and where $L_b(II)$ is the Lie derivative of II given by

$$(L_b(II))_{kl} = b^j h_{kl,j} + b^j_{,k} h_{jl} + b^j_{,l} h_{jk}.$$

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where the Lie derivative of W, $L_b(W)$ is

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$$\frac{\partial}{\partial t}\sqrt{g} = \left(-adH + \nabla\cdot b\right)\sqrt{g}$$

where the divergence and mean curvature are

$$\nabla\cdot b = b^j_{;j}$$

$$H = \frac{1}{d}\mathrm{Trace}\,W.$$

Note that immersions persist as immersions ($g \neq 0$) as long as the evolution is smooth.

Total area, mean curvature

The total area

$$A = \int \sqrt{g} d\alpha = \int_f dS$$

satisfies

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Using

$$\text{Trace } I^{-1} \nabla \nabla a = \Delta_f(a)$$

and taking the trace of the evolution of the Weingarten map we obtain the equation for H

$$\frac{\partial}{\partial t} H = \frac{1}{d} \Big(a \text{Trace}(W^2) + \Delta_f(a) \Big) + b^j H_{,j}.$$

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where $\widetilde{W} = (\text{Trace } W)\text{Id} - W$. We note

$$\frac{\partial}{\partial t}(K\sqrt{g}) = \sqrt{g}\Big[\text{Trace}\left(\widetilde{W}(I^{-1}\nabla\nabla a + L_b(W))\right) + Kb^j_{;j}\Big]$$

$$= \frac{\partial}{\partial \alpha^i}\left(\sqrt{g}g^{ij}\widetilde{W}^k_j \frac{\partial a}{\partial \alpha^k} + b^i K\sqrt{g}\right)$$

verifies the time independence of the Gauss-Bonnet formula $\int_f KdS = \chi(f)$.

Example: d=1, plane curves.

We write $f(\alpha) = z(\alpha)$. Usual differentiation with respect to the only variable (other than time) is denoted by a prime. The Weingarten matrix is simply the curvature κ of the curve z. The Laplace-Beltrami operator Δ_f is the second derivative with respect to arclength. We obtain:

$$\frac{\partial}{\partial t}\kappa = a\kappa^2 + \frac{d^2}{ds^2}a + b\kappa'$$

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We note that the time invariance of the rotation number $\int_f \kappa ds$ follows: the quantity $q = \kappa|z'|$ obeys the conservation law

$$\frac{\partial}{\partial t}q = \left(|z'|^{-1}a' + bq\right)'.$$

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semilinear heat equation. Self-similar blow up, finite time extinction: $\frac{d}{dt}A = -\int_f \kappa^2 ds$, $\int_f \kappa ds = 1$, Schwartz:

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3) Evolution by arclength derivative of curvature: $a = \kappa_s$, $b = 0$. Length (A) is conserved $\frac{d}{dt}A = 0$. Curvature equation= modified KdV:

$$\partial_t \kappa = \kappa^2 \kappa_s + \frac{d^3}{ds^3}\kappa$$

Does not blow up, completely integrable.

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The fluid obeys equations (Navier-Stokes, Euler, Hele-Shaw, Boussinesq, SQG, porous medium, etc). If the interface is passively carried, then a and b are given.

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$$a = n \cdot \nabla p(x,y,t)_{|\,(x,y)=z(\alpha,t)}$$

is the Dirichlet-to-Neumann of $\gamma\kappa$.

Example, $d = 1$: Irrotational inviscid flow

Irrotational 2d Euler flow. Then $v = \nabla\Phi$. Let Ω be the fluid domain and let $f = \partial\Omega$. Bernoulli:

$$\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + p = 0$$

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Computing

$$a(\alpha, t) = n \cdot \nabla\Phi(x, y, t)_{|\,(x,y)=z(\alpha,t)}$$

$$b(\alpha, t) = \frac{1}{|z'(\alpha, t)|^2}\partial_\alpha(\Phi(z(\alpha, t), t))$$

The normal derivative $a = \Lambda\phi$, Dirichlet-to-Neumann, $\phi = \Phi_{|\,f}$. If $\gamma = 0$ problem can be ill posed (Ebin). If $\gamma > 0$, pinchoff computed (Day-Hinch-Lister), but problem largely open.

Slender jets

Axisymmetric Navier-Stokes without swirl, with surface tension and gravity. Variables r, x. Interface:

$$r = h(x, t)$$

Boundary conditions:

$$\left(p\mathbb{I} - \nu\left(\nabla v + \nabla v^T\right)\right) \cdot n = \gamma H n$$

Assume: slender jet, i.e. distances across r much smaller than along x. Eggers-Dupont '94: systematic derivation of equations for $h(x, t)$ and axial velocity $u(x, t)$

$$\partial_t h + u\partial_x h = -\frac{1}{2} h\partial_x u,$$
$$\partial_t u + u\partial_x u + \gamma\partial_x\left(\frac{1}{h}\right) = 3\nu\frac{\partial_x(h^2\partial_x u)}{h^2} - g,$$

Finite time pinchoff, matching experiments (Nagel et al). Viscous forces cannot be neglected at pinchoff. Irrotationality fails.

Compressible degenerate viscous flow, and active potentials

$$\partial_t \rho + \partial_x (u\rho) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) = -\partial_x p(\rho) + \partial_x(\mu(\rho)\partial_x u) + \rho f$$
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Contain: viscous shallow water:

$$p(\rho) = \frac{g}{2}\rho^2 \quad \text{and} \quad \mu(\rho) = 4\nu\rho,$$

Compressible degenerate viscous flow, and active potentials

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Note negative pressure law! Note $\gamma \neq \gamma$!

No singularity without pinchoff

Let $\mathbb{T} = [0, 1]$. We consider periodic boundary conditions.

Theorem

(Drivas, Nguyen, Pasqualotto, C, '18). Let f be smooth enough,

$$f \in L^2(0, T; H^{k-1}(\mathbb{T})),$$

$k \geq 3$, $T > 0$. Assume either one of
A) $c_p > 0$ and $\alpha > \frac{1}{2}, \gamma \neq 1, \ \gamma \geq \alpha - \frac{1}{2}$ (covering viscous shallow water)

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A) $c_p > 0$ and $\alpha > \frac{1}{2}$, $\gamma \neq 1$, $\gamma \geq \alpha - \frac{1}{2}$ (covering viscous shallow water) or
B) $c_p < 0$ and $\frac{1}{2} < \alpha \leq \frac{3}{2}$, $\gamma < 1$, $0 < \gamma \leq \alpha$ (covering Eggers-Dupont equations).
Then solutions (u, ρ) on $[0, T^*)$ satisfy

$$\sup_{T\in[0,T^*)} \|\rho\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)} \|u\|_{L^\infty(0,T;H^k)} + \sup_{T\in[0,T^*)} \|u\|_{L^2(0,T;H^{k+1})} < \infty$$

and can be uniquely continued past T^* if

$$\inf_{t\in[0,T^*)} \min_{x\in\mathbb{T}} \rho(x, t) > 0.$$

Elements of the proof

The proof is technical and uses higher energy metods building on:
Energy

$$e := \frac{1}{2}\rho|u|^2 + \pi(\rho), \qquad \pi(\rho) = \rho \int_{\bar{\rho}}^{\rho} \frac{p(s)}{s^2} ds.$$

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The active potential

$$w = -p(\rho) + \mu(\rho)\partial_x u.$$

If $f = 0$ the force balance equation is

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hence the name. The active potential obeys a nonlinear heat equation with nondegenerate or less degenerate diffusivity $\frac{\mu(\rho)}{\rho}$ than the momentum equation. Bounds for the norms of the active potential are obtained using energy estimates, and used to close higher energy estimates for the momentum and density.

Hele-Shaw

Two dimensional potential flow with surface tension. $\Omega \subset \mathbb{R}^2$, $u = \nabla p$, $f = \partial\Omega$, with

$$\begin{aligned} \Delta p = 0, &\quad \text{in } \Omega, \\ p = \gamma\kappa &\quad \text{at } f \end{aligned}$$

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so

$$\frac{dV}{dt} = \int_{\partial\Omega} \frac{\partial p}{\partial n} dS = 0$$

and

$$\frac{dA}{dt} = -\frac{1}{\gamma}\int_{\partial\Omega} p\frac{\partial p}{\partial n} dS = -\frac{1}{\gamma}\int_{\Omega} |\nabla p|^2 dx < 0$$

Hele-Shaw neck model

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Computations showed self-similar behavior with infinite time pinchoff. Other data lead to finite time pinchoff.

Energy dissipation, steady states

The energy

$$E(h) = \frac{1}{2}\int_I |\partial_x h(x)|^2 dx + P\int_I h(x)dx$$

decays on solutions

$$\frac{d}{dt}E(h(t)) = -D(h(t))$$

where

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The steady solutions:

$$h_P(x) = \frac{P}{2}(x^2 - 1) + 1,$$

if $P \leq 2$ and

$$h_P(x) = \begin{cases} \frac{P}{2}(|x| - x_P)^2, & \text{for } x_P \leq |x| \leq 1, \\ 0, & \text{for } |x| < x_P \end{cases}$$

for $P > 2$, with $x_P = 1 - \sqrt{\frac{2}{P}}$.

Weak solutions, uniqueness and variational characterization

$$\partial_x(h\partial_x^3 h) = \partial_x^2(h\partial_x^2 h - \frac{1}{2}(\partial_x h)^2).$$

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Theorem

(CENV '17) The equation has global weak solutions $h(t)$ which are nonnegative, belong to C^2 near the boundary, satisfy the boundary conditions, and are in $L^2([0,T],H^2(I))$.

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(CENV '17) If $h \geq 0$ and $h \in H^1(I)$ with $h(\pm 1) = 1$ then

$$E(h) \geq E(h_P).$$

Moreover, $E(h) = E(h_P)$ if and only if $h = h_P$.

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Moreover, $E(h) = E(h_P)$ if and only if $h = h_P$.
Let h_n be a sequence of nonnegative $H^3(I)$ functions satisfying the boundary conditions, which are uniformly bounded in $H^1(I)$ and satisfy $\lim_{n\to\infty} D(h_n) = 0$. Then h_n converge weakly in $H^1(I)$ to h_P and strongly in $H^3_{loc}(\{x \mid \ h_P(x) > 0\})$.

Pinchoff

Theorem

(CENV) **1.** *If* $P < 2$ *then* h_P *is asymptotically stable in* $H^1(I)$*:*

$$\|h(t) - h_P\|_{H^1(I)} \leq C\|h_0 - h_P\|_{H^1(I)} e^{-ct}$$

for $\|h_0 - h_P\|_{H^1(I)} \leq \delta$*. Moreover* $h(t)$ *converge to* h_P *in* $H^3(I)$*.*

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for $\|h_0 - h_P\|_{H^1(I)} \leq \delta$. Moreover $h(t)$ converge to h_P in $H^3(I)$.
2. *If $P \geq 2$, then starting from positive $h_0 \in H^3(I)$ the solution pinches off in finite time or in infinite time. If the pinchoff is in infinite time then there exists a sequence of times $t_n \to \infty$ such that $h(t_n)$ converges to h_P weakly in $H^1(I)$ and in $H^3_{loc}(\{x \mid h_P(x) > 0\})$.*

Local existence, blow up= pinchoff

Let

$$X(T) = L^{\infty}([0,T]; H^3(I)) \cap L^2([0,T]; H^5(I))$$

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(CENV '17) Let $h_0 \in H^3(I)$ be a positive initial datum, satisfying the boundary conditions. Let $m(0) = \min_I h_0(x) > 0$. There exists a positive time $T > 0$ depending only on P, $\|h_0\|_{H^3(I)}$ and $m(0)$ such that the problem has a unique solution $h \in X(T)$ which satisfies $m(T) = \inf_{I\times[0,T]} h(x,t) > 0$.

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$$\|h\|_{X(T)} \leq \mathcal{F}(m(T)^{-1}, \|h_0\|_{H^3(I)})$$

holds with $\mathcal{F}$ a continuous increasing function depending only on P.

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Blow up requires $m(T) = 0$. There exists a constant C such that

$$\int_0^T D(h(t))dt \leq C(\|h_0\|_{H^3(I)} + 1)$$

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$$\int_0^T D(h(t))dt \leq C(\|h_0\|_{H^3(I)} + 1)$$

so $T = \infty$ triggers convergence to h_P.

Elements of proof, I

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Elements of the proof II

Linear problem

$$\partial_t h + \partial_x (g \partial_x^3 h) = 0$$

with the same boundary conditions ($h(\pm 1, t) = 1$, $\partial_x^2 h(\pm 1, t) = P$).
Take $m_g = \inf_{I \times [0,T]} g(x,t) > 0$.

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$$\|h\|_{X(T)} \leq \mathcal{F}(m_g^{-1}, \|g\|_{L^\infty(I;H^2(I))}, \|\partial_t g\|_{L^1(I;L^\infty(I))}, \|h_0\|_{H^3(I)})$$

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The active potential

$$w = g \partial_x^3 h$$

obeys

$$w_t = -g \partial_x^4 w + \frac{\partial_t g}{g} w$$

with selfadjoint Neumann-Neumann boundary conditions $\partial_x w(\pm 1, t) = \partial_x^3 w(\pm 1, t) = 0$ which follow from the boundary conditions for $\partial_t h$.

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$$\|h\|_{X(T)} \le \mathcal{F}(m_g^{-1}, \|g\|_{L^\infty(I;H^2(I))}, \|\partial_t g\|_{L^1(I;L^\infty(I))}, \|h_0\|_{H^3(I)})$$

The active potential

$$w = g\partial_x^3 h$$

obeys

$$w_t = -g\partial_x^4 w + \frac{\partial_t g}{g} w$$

with selfadjoint Neumann-Neumann boundary conditions $\partial_x w(\pm 1, t) = \partial_x^3 w(\pm 1, t) = 0$ which follow from the boundary conditions for $\partial_t h$. The active potential has therefore very good energy bounds, if $g > 0$ and $\partial_t g$ is no too bad. Approximations, bootstraps, high energy bounds...

Elements of the proof II

Linear problem

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Inviscid limit

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3D result

Theorem

(Vicol, C, '17) Let u_n be a sequence of weak solutions of the Navier-Stokes equations

$$\partial_t u_n + u_n \cdot \nabla u_n - \nu_n \Delta u_n + \nabla p_n = f_n$$

in bounded domain $\Omega \subset \mathbb{R}^3$, with $\nabla \cdot u_n = 0$, and f_n bounded in $L^2(0, T; L^2(\Omega))$, converging weakly to f, with $u_n(0)$ divergence-free and bounded in $L^2(\Omega)$ and $\nu_n \to 0$.

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$$\sup_n \int_0^T \int_K |u_n(x+y,t) - u_n(x,t)|^2 dxdt \leq E_K |y|^{\zeta_2}$$

holds for $|y| < dist(K, \partial\Omega)$ in the inertial range

$$|y| \geq \eta(n), \quad \text{with} \quad \lim_{n\to\infty} \eta(n) = 0.$$

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Let $u_n(t)$ converge weakly in $L^2(\Omega)$ to $u_\infty(t)$ for almost all $t \in (0, T)$. Then u_∞ is a weak solution of the Euler equations.

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3. It is possible to remove the assumption of almost all time $L^2(\Omega)$ convergence, and replace it with the weak convergence in $L^2(0, T; L^2(\Omega))$, (Theo Drivas, Huy Nguyen, '18).
4. The result means that any reasonable turbulence scaling assumptions away from boundaries imply weak Euler limit.

2D Result

Theorem

(Vicol, C, '17) Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary. Let u_n be a sequence of solutions of Navier-Stokes equations with viscosities $\nu_n \to 0$, driven by forces $f_n \in H^1(\Omega)$ that are uniformly bounded in $H^1(\Omega)$ and converge weakly in $H^1(\Omega)$ to f. We take divergence free initial data $u_n(0)$ belonging to $H_0^1(\Omega)$ and uniformly bounded in $L^2(\Omega)$. Assume that for any $K \subset\subset \Omega$,

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uniformly in n. Then any weak limit u_∞ in $L^2(0, T; L^2(\Omega))$ of the sequence u_n, is a weak solution of the Euler equations

$$\partial_t \omega_\infty + u_\infty \cdot \nabla \omega_\infty = g = \nabla^\perp \cdot f$$

with $\omega_\infty = \nabla^\perp \cdot u_\infty$.

2D result, continued

The solution has bounded energy,

$$u_\infty \in L^\infty(0, T; L^2(\Omega)).$$

and for any compact $K \subset\subset \Omega$,

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2. Condition of uniform local interior enstrophy bound follows from uniform gradient bounds $\sup_t \|\nabla\phi\|^2_{L^\infty(\Omega)}$ for

$$\partial_t\phi + u\cdot\nabla\phi + \nu\Delta\phi = 0,$$

with final condition

$$\phi(\tau) = \mathbf{1}_K.$$

Stochastic interpretation.

2D with vortex sheet data

Theorem

(Nussenzveig, Lopes, Vicol, C, 2018) Let ν_n *be positive numbers such that* $\nu_n \to 0$*. Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with smooth boundary. Let* $\{u_0^n\}_n \subset L^2(\Omega)$ *and let* $u^n \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$ *solve 2DNSE with viscosity* $\nu = \nu_n$*, no slip boundary condition and initial data* u_0^n*. Let* $\omega^n = \omega^n(t,\cdot) = \nabla^\perp \cdot u^n(t,\cdot)$*. Let* u^∞ *be such that* $u^n \rightharpoonup u^\infty$ *weak-*$*$ $L^\infty(0,T; L^2(\Omega))$*.*

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1. $\{\omega^n\} \subset L^\infty((0,T);L^1_{loc}(\Omega))$ and, for each $K \subset\subset \Omega$, there exists $C_K > 0$ so that

$$\sup_n \sup_{t\in(0,T)} \|\omega^n(t,\cdot)\|_{L^1(K)} \leq C_K < \infty;$$

2. For any $K \subset\subset \Omega$ we have

$$\sup_n \int_0^T \left(\sup_{x\in K} \int_{B(x;r)\cap\Omega} |\omega^n(t,y)|\, dy \right) dt \to 0 \text{ as } r \to 0.$$

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Then u^∞ is a weak solution of the incompressible Euler equations.

Idea of proof for 2D with vortex sheet

Weak distributional solution of 2D Euler, in our case gives in vorticity:

$$\omega \in L^\infty(0, T; H^{-1}(\Omega) \cap M_{loc}(\Omega))$$

where $M_{loc}(\Omega)$ are signed measures with uniform local mass.

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$$H_\phi(x,y,t) = \frac{1}{2}\left(K(x,y)\nabla\phi(x,t) + K(y,x)\nabla\phi(y,t)\right)$$

and $K(x,y) = \nabla_x^\perp G_\Omega(x,y)$ the Biot-Savart kernel and $\chi \in C_0^\infty(\Omega)$ identically one on the support of ϕ.

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The assumption takes care of the diagonal, only place where we do not have continuity in H_ϕ.

Concluding Remarks

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- Drop and slender jet pinchoff are highly nonlinear, nonlocal problems with geometric flavor, but not geometric problems. They are largely open.
- Vanishing of the dissipation rate follows from weak convergence in $L^2(\Omega)$ for all times (only) if the Euler equation limit is conservative. We proved results of emergence of weak, possibly dissipative solutions of Euler equations in 3D if the ensemble of Navier-Stokes solutions obeys a local-in-space but uniform in the ensemble second order structure function scaling from above. In two dimensions, we proved the emergence of weak solutions form arbitrary families of strong solutions of Navier-Stokes equations with uniform interior local vorticity measure bounds which allow the formation of vortex sheets in the limit.