SQG in Bounded Domains

Peter Constantin

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Collaborators

- Mihaela Ignatova, (Princeton)
- Huy Nguyen (Princeton)

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Blow up problem open: 3D Euler, 2D SQG, 2D Boussinesq, 2D incompressible porous medium, 2D Oldroyd B. Similar.

Numerical results

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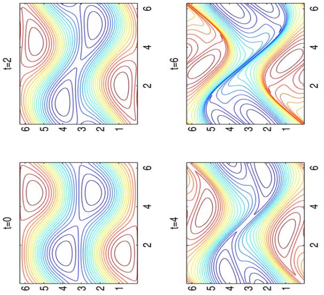
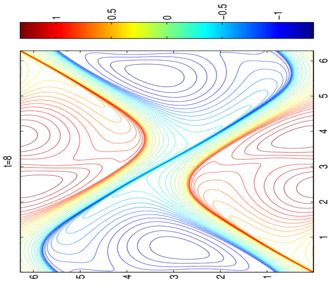
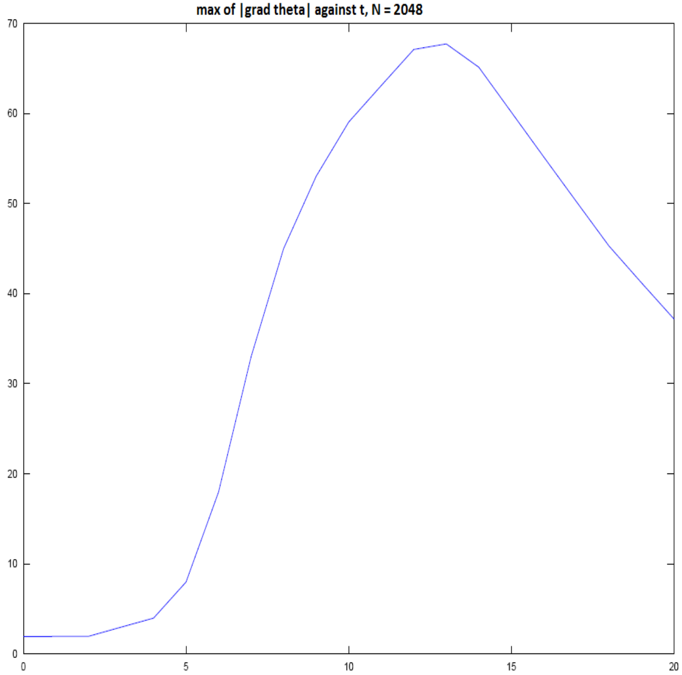
C-Lai-Sharma-Tseng-Wu. Parallel computation, cluster of 128 machines, well resolved for long time. Same initial data.


t=0
t=2
t=4
t=6



t=8
1
0.5
0
−0.5
−1
1
2
3
4
5
6


max of |grad theta| against t, N = 2048
70
60
50
40
30
20
10
0
0
5
10
15
20

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Weak continuity:

$$\|(-\Delta)^{-1}\left[N(\theta_1,\theta_1) - N(\theta_2,\theta_2)\right]\|_w \le$$
$$C\left\{\|\theta_1-\theta_2\|_w\left(1+\log_+\|\theta_1-\theta_2\|_w\right)\right\}\left(\|\theta_1\|_{L^2}+\|\theta_2\|_{L^2}\right)$$

with $\|f\|_w = \sup_{j\in\mathbb{Z}^2}\left|\widehat{f}(j)\right|$.

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- quasilinear, critical in the sense of Goldilocks: easy for Λ^s, $s > 1$, hard for $s < 1$.)

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Multiply by $\nabla^\perp \theta$ to have positive quantities:

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$|\nabla u| \sim q$: Q is cubic. Nonlinear lower bound ! (Vicol, C)

$$D(q) = q\Lambda q - \frac{1}{2}\Lambda\left(q^2\right) \ge \frac{q^3}{\|\theta\|_{L^\infty}}$$

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- Smallness of α: The term corresponding to Q in the finite difference version of the argument has a small (α) prefactor and it is dominated by the term corresponding to $D(q)$

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- Global existence of solutions for critical dissipative SQG: global interior Lipschitz bounds (Ignatova, C)
- Global L^2 weak solutions for inviscid SQG (H.Q. Nguyen, C)

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Theorem

(C, Ignatova) Let $\theta(x,t)$ be a smooth solution of critical SQG in the smooth bounded domain Ω. There exists $0 < \alpha < 1$ depending only on $\|\theta_0\|_{L^\infty(\Omega)}$ and Ω, and a constant $\Gamma > 0$ depending only on the domain Ω (in particular: not on T) such that

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$$\sup_{0\le t<T} \|\theta(t)\|_{C^\alpha(\Omega)} \le \Gamma \|\theta_0\|_{C^\alpha(\Omega)}.$$

Moreover,

$$\sup_{x\in\Omega, 0\le t<T} d(x)|\nabla_x\theta(x,t)| \le \Gamma_1\left[\sup_{x\in\Omega} d(x)|\nabla_x\theta_0(x)| + P\left(\|\theta_0\|_{L^\infty(\Omega)}\right)\right]$$

Elements of the proof

$$[f]_\alpha = \sup_{x\in\Omega}(d(x))^\alpha \left(\sup_{h\neq 0, |h|<d(x)} \frac{|f(x+h)-f(x)|}{|h|^\alpha} \right) < \infty.$$

$d(x) = dist(x, \partial\Omega))$. Norm in $C^\alpha(\Omega)$ (interior)

$$\|f\|_{C^\alpha} = \|f\|_{L^\infty(\Omega)} + [f]_\alpha.$$

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- Gaussian bounds for heat kernel; cancellation due to translation invariance effective for small time.

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- Gaussian bounds for heat kernel; cancellation due to translation invariance effective for small time.
- Nonlinear maximum principle (lower bound for Λ_D) giving smoothing and a strong boundary repulsion damping effect.

Elements of the proof

$$[f]_\alpha = \sup_{x\in\Omega}(d(x))^\alpha \left(\sup_{h\neq 0, |h|<d(x)} \frac{|f(x+h)-f(x)|}{|h|^\alpha}\right) < \infty.$$

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- Finite difference bounds for Riesz transforms using the nonlinear max principle bound in its finite difference variant.

Basics in bounded domains

- $\Omega \subset \mathbb{R}^d$ open, bounded, smooth boundary
- $-\Delta$ Laplacian operator with homogeneous Dirichlet boundary conditions
- w_j are $L^2(\Omega)$ - normalized eigenfunctions, λ_j corresponding eigenvalues counted with their multiplicities

$$-\Delta w_j = \lambda_j w_j$$

- $0 < \lambda_1 \leq \cdots \leq \lambda_j \to \infty$
- $-\Delta$ positive self-adjoint operator in L^2 with domain

$$\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$$

- The ground state is positive and

$$c_0 d(x) \leq w_1(x) \leq C_0 d(x)$$

for all $x \in \Omega$, where

$$d(x) = dist(x, \partial\Omega)$$

Fractional powers in terms of heat kernel

$$(-\Delta)^{\alpha} f = \sum_{j=1}^{\infty} \lambda_j^{\alpha} f_j w_j$$

$f_j = \int_{\Omega} f(y) w_j(y)\, dy$

$$\Lambda_D = (-\Delta)^{\frac{1}{2}}$$

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for $f \in \mathcal{D}((-\Delta)^{\alpha})$.

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$$\lambda^{\alpha} = c_{\alpha} \int_0^{\infty} (1 - e^{-t\lambda}) t^{-1-\alpha}\, dt$$

Gaussian bounds for the heat kernel

$$(e^{t\Delta}f)(x) = \int_\Omega H_D(t,x,y)f(y)dy$$

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Davies '87, Zhang '02, '06: There exists a time $T > 0$ depending on the domain Ω and constants c, C, k, K, depending on T and Ω such that

$$\min\left(\frac{w_1(x)}{|x-y|},1\right)\min\left(\frac{w_1(y)}{|x-y|},1\right)t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{kt}} \leq H_D(t,x,y)$$
$$\leq C\min\left(\frac{w_1(x)}{|x-y|},1\right)\min\left(\frac{w_1(y)}{|x-y|},1\right)t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{Kt}}$$

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$$\frac{|\nabla_x H_D(t,x,y)|}{H_D(t,x,y)} \le C\begin{cases} \frac{1}{d(x)}, & \text{if } \sqrt{t} \ge d(x),\\ \frac{1}{\sqrt{t}}\left(1+\frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \le d(x)\end{cases}$$

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holds for all $0 \le t \le T$. Interchange x and y:

$$\partial_1^\beta H_D(t,y,x) = \partial_2^\beta H_D(t,x,y) = \sum_{j=1}^\infty e^{-t\lambda_j}\partial_y^\beta w_j(y)w_j(x).$$

Additional bounds; translation invariance effect

$$|\nabla_x \nabla_x H_D(x, y, t)| \le C t^{-1-\frac{d}{2}} e^{-\frac{|x-y|^2}{\tilde{K} t}}$$

holds for $t \le c d(x)^2$ and $0 < t \le T$.

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valid for $t \le cd(x)^2$. nonsingular at $x = y$! These bounds reflect the fact that translation invariance is remembered in the solution of the heat equation with Dirichlet boundary data for short time, away from the boundary. They are essential in the proof of bounds for commutators with differentiation.

The convex damping inequality

Proposition

(C, Ignatova) Let Ω *be a bounded domain with smooth boundary, let* $0 < s < 2$*. There exists a constant C depending on the domain and on s such that for every* Φ*, a* C^2 *convex function satisfying* $\Phi(0) = 0$*, and every* $f \in C_0^\infty(\Omega)$

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$$\Phi'(f)\Lambda_D^s f - \Lambda_D^s(\Phi(f)) \geq \frac{C}{d(x)^s}\left(f(x)\Phi'(f(x)) - \Phi(f(x))\right)$$

holds pointwise in Ω.

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Dramatically different from $\mathbb{R}^d$!

The nonlinear bound for derivatives

Theorem

(C, I) Let $f \in L^\infty(\Omega) \cap \mathcal{D}(\Lambda_D^s)$, $0 \le s < 2$. *Assume that* $f = \partial q$ *with* $q \in L^\infty(\Omega)$ *and* ∂ *a first order derivative. Then there exist constants* c, C *depending on* Ω *and* s *such that*

$$f \Lambda_D^s f - \frac{1}{2}\Lambda_D^s f^2 \ge c\|q\|_{L^\infty}^{-s}|f_d|^{2+s}$$

holds pointwise in Ω, *with*

$$|f_d(x)| = \begin{cases} |f(x)| & \text{if } |f(x)| \ge C\|q\|_{L^\infty(\Omega)}\frac{1}{d(x)}, \\ 0 & \text{if } |f(x)| \le C\|q\|_{L^\infty(\Omega)}\frac{1}{d(x)}, \end{cases}$$

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Proof: nontrivial, uses precise bounds on the heat kernel and

$$f\Lambda_D^s f - \frac{1}{2}\Lambda_D^s f^2 \ge \frac{c_s}{2}\int_0^\infty t^{-1-\frac{s}{2}}dt\int_\Omega H_D(t,x,y)(f(x)-f(y))^2 dy$$

Good cutoff

Lemma

(C,I) Let Ω *be a bounded domain with* C^2 *boundary. For* $\ell > 0$ *small enough (depending on* Ω*) there exist cutoff functions* $\chi_\ell = \chi$ *with the properties:* $0 \leq \chi \leq 1$, $\chi(y) = 0$ *if* $d(y) \leq \frac{\ell}{4}$, $\chi(y) = 1$ *for* $d(y) \geq \frac{\ell}{2}$, $|\nabla^k \chi| \leq C\ell^{-k}$ *with* C *independent of* ℓ *and*

$$\int_\Omega \frac{(1-\chi(y))}{|x-y|^{d+j}} dy \leq C \frac{1}{d(x)^j}$$

and

$$\int_\Omega |\nabla \chi(y)| \frac{1}{|x-y|^d} \leq C \frac{1}{d(x)}$$

hold for $j \geq 0$ *and* $d(x) \geq \ell$. *We will refer to such* χ *as a "good cutoff".*

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hold for $j \ge 0$ and $d(x) \ge \ell$. We will refer to such χ as a "good cutoff".

Useful because of the Gaussian bounds on the heat kernel. Makes work in Ω look like work in half-space, where $\chi_\ell = \chi_1(\frac{x_d}{\ell})$, without changing coordinates.

Nonlinear bound, finite differences

Theorem

(C,I) Let Ω be a bounded domain with smooth boundary. Let $\chi \in C_0^\infty(\Omega)$ be a good cutoff with scale $\ell > 0$ and let

$$f(x) = \chi(x)(\delta_h q(x)) = \chi(x)(q(x+h) - q(x))$$

with $q \in L^\infty(\Omega) \cap H_0^1(\Omega)$. Then

$$D(f)(x) = (f\Lambda_D f)(x) - \frac{1}{2}(\Lambda_D f^2)(x) \geq \gamma_1 |h|^{-1} \frac{|f_d(x)|^3}{\|q\|_{L^\infty}} + \gamma_1 \frac{f^2(x)}{d(x)}$$

holds a.e. pointwise in Ω when $|h| \leq \frac{\ell}{16}$, and $d(x) \geq \ell$ with

$$|f_d(x)| = |f(x)|, \qquad \text{if } \ |f(x)| \geq M\|q\|_{L^\infty(\Omega)} \frac{|h|}{d(x)}.$$

Commutator

Let χ be a good cutoff.

Lemma

(C,I) There exists a constant Γ_0 *such that the commutator*

$$C_h(\theta) = \chi\delta_h\Lambda_D\theta - \Lambda_D(\chi\delta_h\theta)$$

obeys

$$|C_h(\theta)(x)| \leq \Gamma_0 \frac{|h|}{d(x)^2} \|\theta\|_{L^\infty(\Omega)}$$

for $d(x) \geq \ell$, $|h| \leq \frac{\ell}{16}$.

Finite difference of Riesz transform

Lemma

(C,I) Let χ be a good cutoff, and let u be defined by

$$u = R_D^{\perp}\theta.$$

Then

$$|\delta_h u(x)| \leq C\left(\sqrt{\rho D(f)(x)} + \|\theta\|_{L^\infty}\left(\frac{|h|}{d(x)} + \frac{|h|}{\rho}\right) + |\delta_h \theta(x)|\right)$$

holds for $d(x) \geq \ell$, $\rho \leq cd(x)$, $f = \chi\delta_h\theta$ and with C a constant depending on Ω.

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holds for $d(x) \geq \ell$, $\rho \leq cd(x)$, $f = \chi\delta_h\theta$ and with C a constant depending on Ω.

This gives a bound on $|h|^{-1}|\delta_h u(x)|$ which costs $D(f)$.

Idea of proof of Hölder bound

Good cutoff, and equation for $\delta_h\theta$ imply:

$$\frac{1}{2}L_{\chi}\left(\delta_h\theta\right)^2 + D(f) + (\delta_h\theta)C_h(\theta) = 0$$

with

$$L_{\chi}g = \partial_t g + u\cdot\nabla_x g + \delta_h u\cdot\nabla_h g + \Lambda_D(\chi^2 g).$$

and

$$D(f) \geq \gamma_1|h|^{-1}\|\theta\|_{L^\infty}^{-1}|(\delta_h\theta)_d|^3 + \gamma_1(d(x))^{-1}|\delta_h\theta|^2$$

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Multiply by $|h|^{-2\alpha}$ with $\epsilon = \alpha\|\theta_0\|_{L^\infty}$ small.

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Multiply by $|h|^{-2\alpha}$ with $\epsilon = \alpha\|\theta_0\|_{L^\infty}$ small. Obtain:

$$L_\chi\left(\frac{\delta_h\theta(x)^2}{|h|^{2\alpha}}\right) + \frac{\gamma_1}{4d(x)}\left(\frac{\delta_h\theta(x)^2}{|h|^{2\alpha}} - \Gamma_1\ell^{-2\alpha}\|\theta\|_{L^\infty}^2\right) \leq 0.$$

Inviscid global weak solutions, bounded domains

Theorem

(C, Q.H. Nguyen.) *Let $\theta_0 \in L^2(\Omega)$. There exists a weak solution of inviscid SQG*

$$\partial_t \theta + R_D^{\perp}\theta \cdot \nabla \theta = 0$$

with $\psi = \Lambda_D^{-1}\theta \in C([0,\infty), H_0^{1-\epsilon}(\Omega))$ for any $0 < \epsilon < 1$. The Hamiltonian

$$\int_\Omega \theta(t)\Lambda_D^{-1}\theta(t)dx$$

is conserved in time, and the $L^2(\Omega)$ norm of $\theta(t)$ is nonincreasing in time.

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Theorem

(C, Ignatova, Nguyen) Let $T > 0$ and let $\theta_k(x,t)$, $0 \le t \le T$ be a sequence of solutions of critical SQG with "viscosities" $\nu_k \to 0$ and initial data uniformly bounded in $L^2(\Omega)$. Then the limit of any weakly L^2 convergent subsequence is a weak solution of inviscid SQG.

Elements of Proof

Weak continuity from commutator structure (adapted for bounded domains): ϕ test function, $\psi = \Lambda_D^{-1}\theta$:

$$\int_\Omega (R_D^\perp\theta \cdot \nabla\theta)\phi dx$$
$$= -\frac{1}{2}\int_\Omega \psi[\Lambda_D, \nabla^\perp]\psi \cdot \nabla\phi dx + \frac{1}{2}\int_\Omega \nabla^\perp\psi \cdot [\Lambda_D, \nabla\phi]\psi dx$$

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Together with commutator estimates

Theorem

(Ignatova, C) Let $\chi \in B(\Omega)$ with $B(\Omega) = W^{2,\infty}(\Omega) \cap W^{1,\infty}(\Omega)$ if $d \geq 3$, and $B(\Omega) = W^{2,p}(\Omega)$ with $p > 2$ if $d = 2$. There exists a constant $C = C(d, p, \Omega)$ such that

$$\|\Lambda_D^{\frac{1}{2}}[\Lambda_D, \chi]\psi\|_{L^2(\Omega)} \leq C\|\chi\|_{B(\Omega)}\|\Lambda_D^{\frac{1}{2}}\psi\|_{L^2(\Omega)}.$$

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Together with commutator estimates

Theorem

(Ignatova, C) Let $\chi \in B(\Omega)$ with $B(\Omega) = W^{2,\infty}(\Omega) \cap W^{1,\infty}(\Omega)$ if $d \geq 3$, and $B(\Omega) = W^{2,p}(\Omega)$ with $p > 2$ if $d = 2$. There exists a constant $C = C(d, p, \Omega)$ such that

$$\|\Lambda_D^{\frac{1}{2}}[\Lambda_D, \chi]\psi\|_{L^2(\Omega)} \leq C\|\chi\|_{B(\Omega)}\|\Lambda_D^{\frac{1}{2}}\psi\|_{L^2(\Omega)}.$$

Theorem

(Ignatova, Nguyen, C.) For $1 \leq p \leq \infty$, $0 < s < 2$, there exists C such that for all $x \in \Omega$

$$|[\Lambda_D^s, \nabla]\psi(x)| \leq Cd(x)^{-1-s-\frac{d}{p}}\|\psi\|_{L^p(\Omega)}$$

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