Derivation of confined non-local diffusion equations from kinetic models

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Outline

The fractional heat equation

The fractional heat equation reads for $s \in (0,1)$:

$$\begin{cases} \partial_t \rho = -\big(-\Delta\big)^s \rho & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ \rho(0,x) = \rho_{in}(x) & x \in \mathbb{R}^d \end{cases}$$

Underlying stochastic process: $2s$-stable symmetric Lévy process.
The fractional Laplacian can be defined as a singular integral

$$\big(-\Delta\big)^s \rho(x) = c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\rho(x)-\rho(y)}{|x-y|^{d+2s}} \, \mathrm{d}y.$$

The kinetic model

$f(t,\cdot,\cdot)$ is the probability density of a cloud of particles, we consider for its evolution a linear kinetic model

$$\partial_t f + v \cdot \nabla_x f = L(f)$$

where L is a linear collision operator. We will consider either the fractional Fokker-Planck operator:

$$L(f) := \nabla_v \cdot (vf) - \big(-\Delta_v\big)^s f$$

whose velocity equilibrium distribution F is a heavy tailed distribution

$$F(v) \underset{|v| \gg 1}{\sim} \frac{1}{|v|^{d+2s}}.$$

Or the Linear Relaxation operator with the same velocity equilibrium:

$$L(f) = \rho_f F - f \quad \text{with } \rho_f = \int_{\mathbb{R}^d} f \, \mathrm{d}v$$

Outline of the moment method in $\mathbb{R}^d$

1) Anomalous rescaling: using the Knudsen number ε

$$\begin{cases} \varepsilon^{2s}\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = L(f_\varepsilon) \\ f^\varepsilon(0,x,v) = f_{in}(x,v) \end{cases}$$

2) A priori estimate: control the quadratic entropy of f^ε to show

$$f^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \rho(t,x)F(v) \text{ weakly in } L^\infty([0,T);L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d)).$$

Note that ρ is actually the limit of the macroscopic densities

$$\rho_\varepsilon := \int_{\mathbb{R}^d} f_\varepsilon \,\mathrm{d}v \underset{\varepsilon\to 0}{\rightharpoonup} \rho(t,x) \text{ weakly in } L^\infty([0,T);L^2(\mathbb{R}^d)).$$

Outline of the moment method in $\mathbb{R}^d$

3) Auxiliary problem: f_ε is a distributional solution of the kinetic model, i.e. for all $\phi \in \mathcal{D}([0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$:

$$\iiint f_\varepsilon \Big(\varepsilon^{2s} \partial_t \phi + \varepsilon v \cdot \nabla_x \phi - L^* \phi \Big) \, \mathrm{d}\mu + \varepsilon^{2s} \iint f_{in} \phi \, \mathrm{d}x \, \mathrm{d}v = 0$$

From $\psi \in \mathcal{D}([0, T) \times \mathbb{R}^d)$ construct a test function ϕ^ε as a solution to: for fractional Fokker-Planck:

$$\begin{cases} \varepsilon v \cdot \nabla_x \phi_\varepsilon - v \cdot \nabla_v \phi_\varepsilon = 0 \\ \phi_\varepsilon(t, x, v = 0) = \psi(t, x) \end{cases}$$

and for Linear Relaxation

$$\phi^\varepsilon - \varepsilon v \cdot \nabla_x \phi^\varepsilon = \psi$$

Outline of the moment method in $\mathbb{R}^d$

4) Take the limit in the weak formulation of the kinetic equation.

$$\iiint f_\varepsilon\Big(\varepsilon^{2s}\partial_t\phi_\varepsilon + \varepsilon v\cdot\nabla_x\phi_\varepsilon - L^*\phi_\varepsilon\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$
$$+\,\varepsilon^{2s}\iint f_{in}(x,v)\phi_\varepsilon(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0$$

becomes, in the limit as ε goes to 0:

$$\iint \rho\Big(\partial_t\psi - (-\Delta)^s\psi\Big)\,\mathrm{d}t\,\mathrm{d}x + \int \rho_{in}\psi(0,x)\,\mathrm{d}x = 0$$

which is the variational formulation of the fractional heat equation.

Kinetic equation on a bounded domain

Consider Ω a smooth bounded domain and define the oriented set $\Sigma_\pm := \big\{(x,v) \in \partial\Omega \times \mathbb{R}^d : \pm v \cdot n(x) > 0\big\}$. Boundary conditions for kinetic equations take the form of a balance between the trace of f over Σ_+ and Σ_-:

- Absorption (or zero inflow) boundary condition

$$\gamma_- f(t,x,v) = 0$$

- Specular reflection boundary condition

$$\gamma_- f(t,x,v) = \gamma_+ f\Big(t, x, v - 2\big(v \cdot n(x)\big) n(x)\Big)$$

- Diffusive boundary condition ($s > 1/2$)

$$\gamma_- f(t,x,v) = c_0 F(v) \int_{w \cdot n(x) > 0} \gamma_+ f(t,x,w) |w \cdot n(x)| \, \mathrm{d}w$$

Absorption boundary condition

If we consider the absorption boundary condition $\gamma_- f(t,x,v) = 0$, we need to add the associated boundary condition to the auxiliary problem which becomes, in the fractional Fokker-Planck case:

$$\begin{cases} \varepsilon v \cdot \nabla_x \phi_\varepsilon - v \cdot \nabla_v \phi_\varepsilon = 0 & \text{in } \mathbb{R}^+ \Omega \times \mathbb{R}^d \\ \phi_\varepsilon(t,x,v=0) = \psi(t,x) & \text{in } \mathbb{R}^+ \times \Omega \\ \gamma_+ \phi_\varepsilon(t,x,v) = 0 & \text{on } \mathbb{R}^+ \times \Sigma_+. \end{cases}$$

If Ω is convex and $\psi \in \mathcal{D}([0,+\infty) \times \Omega)$ we see that a solution to this auxiliary problem is

$$\phi_\varepsilon(t,x,v) = \psi(t,x+\varepsilon v).$$

Anomalous diffusion limit, non-local Dirichlet condition

If we consider the absorption boundary condition $\gamma_- f(t,x,v) = 0$ then we get:

Theorem (C.(18))

Let Ω be convex domain. The limit ρ of f^ε is the unique weak solution to

$$\begin{cases} \partial_t \rho + \big(-\Delta\big)^s \rho = 0 & \textit{in } (0,T)\times\Omega, \\ \rho(0,x) = \rho_{in}(x) & \textit{in } \Omega, \\ \rho(t,x) = 0 & \textit{in } (0,T)\times\big(\mathbb{R}^d\setminus\Omega\big). \end{cases}$$

other reference: Aceves-Sanchez, Schmeiser (17): Linear Boltzmann case in a non-convex domain.

Specular reflection boundary condition

With specular reflections, the auxiliary problem becomes, in the fractional Fokker-Planck case:

$$\begin{cases} \varepsilon v \cdot \nabla_x \phi_\varepsilon - v \cdot \nabla_v \phi_\varepsilon = 0 & \text{in } \mathbb{R}^+ \Omega \times \mathbb{R}^d \\ \phi_\varepsilon(t, x, v = 0) = \psi(t, x) & \text{in } \mathbb{R}^+ \times \Omega \\ \gamma_+ \phi_\varepsilon(t, x, v) = \gamma_- \phi_\varepsilon(t, x, \mathcal{R}_x v) & \text{on } \mathbb{R}^+ \times \Sigma_+. \end{cases}$$

This problem is intimately linked with the free transport equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0 & \text{in } \mathbb{R}^+ \Omega \times \mathbb{R}^d \\ \gamma_+ f(t, x, v) = \gamma_- f(t, x, \mathcal{R}_x v) & \text{on } \mathbb{R}^+ \times \Sigma_+. \\ f(t = 0, x, v) = f_{in}(x, v) & \text{in } \Omega \times \mathbb{R}^d \end{cases}$$

Flow of free transport

Let Ω be a convex domain and define $\eta : \Omega \times \mathbb{R}^d \mapsto \bar{\Omega}$ the associated flow of free transport in the sense that for $x \in \Omega$ and $v \in \mathbb{R}^d$: $\eta(x, v)$ is the end point of the trajectory of free transport starts at x, is specularly reflected upon hitting the boundary, and stops at "time 1", when the length of the trajectory is $|v|$.

Then, we can construct of solution to the auxiliary problem as

$$\phi_\varepsilon(t, x.v) = \psi\big(t, \eta(x, \varepsilon v)\big).$$

Note that is the trajectory never hits the boundary, then $\eta(x, \varepsilon v) = x + \varepsilon v$ and we recover the solution of the auxiliary problem in the whole space.

Anomalous diffusion limit, specular reflection condition

Theorem (C. (18), fVFP case)

If Ω is a half-space or a ball in $\mathbb{R}^d$ then the limit ρ of f^ε is the unique weak solution to

$$\begin{cases} \partial_t \rho + (-\Delta)^s_{SR} \rho = 0 & \text{in } (0, T) \times \Omega \\ \rho(0, x) = \rho_{in}(x) \end{cases}$$

where

$$(-\Delta)^s_{SR} \psi(x) := c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\psi(x) - \psi\big(\eta(x, w)\big)}{|w|^{d+2s}} \, \mathrm{d}w$$

with η the flow of the free transport equation with specular reflection.

Diffusive boundary condition

With the diffusive boundary condition, the auxiliary problem becomes, in the linear relaxation case:

$$\begin{cases} \phi^\varepsilon - \varepsilon v \cdot \nabla_x \phi^\varepsilon = \psi(t,x) \\ \gamma_+ \phi_\varepsilon(t,x,v) = c_0 \displaystyle\int_{w\cdot n(x)<0} \gamma_- \phi_\varepsilon(t,x,w) F(w) |w \cdot n(x)| \,\mathrm{d}w \end{cases}$$

In the whole space, the transport part of this problem can be solved by inverting the operator $\mathrm{Id} - \varepsilon v \cdot \nabla_x$:

$$\phi_\varepsilon(t,x,v) = \int_0^{+\infty} e^{-z} \psi(t, x + \varepsilon z v) \,\mathrm{d}z.$$

Combined with the diffusive boundary, this leads to the following condition on ψ on the boundary: for $x \in \partial\Omega$:

$$\int_{w\cdot n<0} \int_0^{+\infty} e^{-z} F(w) \big[\psi(x + \varepsilon z w) - \psi(x)\big] |w \cdot n| \,\mathrm{d}z \,\mathrm{d}w = 0$$

A non-local gradient of order $2s-1$

In order to write the limit problem for the diffusive boundary condition we define the non-local gradient D^{2s-1}, with $s > 1/2$, as

$$D^{2s-1}[u](x) := C \int_\Omega (y-x)\cdot \nabla u(y) \frac{y-x}{|y-x|^{d+2s}} \,\mathrm{d}y$$

where the constant C depends on F and ν.
Note that if $d=1$ and $\Omega = \mathbb{R}$ then

$$D^{2s-1}u(x) = C \int_{\mathbb{R}} \frac{u'(y)}{|x-y|^{d-2(1-s)}} \,\mathrm{d}y = \widetilde{C}(-\Delta)^{-(1-s)}u'(x).$$

Anomalous diffusion limit, diffusive boundary condition

If we consider the diffusive boundary condition
$\gamma_- f(t,x,v) = c_0 F(v) \int_{w\cdot n(x)>0} \gamma_+ f(t,x,w)|w\cdot n(x)|\,\mathrm{d}w$ with $s > 1/2$
then we get:

Theorem (C., Mellet, Puel (18), Linear Boltzmann case)

If Ω is a half-space then the limit ρ of f^ε is a distributional solution to

$$\begin{cases} \partial_t \rho - divD^{2s-1}[\rho] = 0 & in\ (0,+\infty)\times\Omega, \\ D^{2s-1}[\rho]\cdot n(x) = 0 & on\ (0,+\infty)\times\partial\Omega, \\ \rho(0,x) = \rho_{in}(x) & in\ \Omega. \end{cases}$$

Note that the diffusion operator $\mathcal{L} := \mathrm{div} D^{2s-1}$ can be written as

$$\mathcal{L}[\rho](x) = c_{F,\nu} P.V. \int_\Omega \nabla\rho(y)\cdot\frac{y-x}{|y-x|^{d+2s}}\,\mathrm{d}y.$$

Thank you for your attention !