Lipschitz regularization for fractional mean curvature flow.

Stephen Cameron
University of Chicago

December 13, 2018

Background

Given a bounded smooth set $E \subseteq \mathbb{R}^d$, we can define its s-fractional perimeter to be

$$P_s(E) := [\mathbb{1}_E]_{\dot{W}^{s,1}} = s(1-s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathbb{1}_E(X) - \mathbb{1}_E(Y)|}{|X-Y|^{d+s}} dXdY \\ = s(1-s) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbb{1}_E(X)\mathbb{1}_{CE}(Y)}{|X-Y|^{d+s}} dXdY. \tag{1}$$

for $0 < s < 1$. The s-fractional perimeter interpolates between our notions of perimeter and Lebesgue measure, with

$$P_s(E) \leq C|E|^{1-s} P(E)^s. \tag{2}$$

Background

If E locally minimizes the s-fractional perimeter in some domain Ω, then we call E an s-minimal surface. It then solves the Euler-Lagrange equation

$$H_s(X, E) = s(1-s)P.V. \int_{\mathbb{R}^d} \frac{\mathbb{1}_{CE}(Y) - \mathbb{1}_E(Y)}{|X-Y|^{d+s}} dY = 0, \qquad X \in \partial E \cap \Omega. \tag{3}$$

where H_s is the s-fractional mean curvature. As $s \to 1$, P_s, H_s converge to classical perimeter and mean curvature.

Background

Given a smooth set $E_0 \subseteq \mathbb{R}^d$, we can define its s-fractional mean curvature flow by

$$\partial_t X(t) = -H_s(X(t), E_t)\nu(X(t)), \qquad X(t) \in \partial E_t. \tag{4}$$

In the case that $\partial E_t = \text{graph}(u(t,\cdot) : \mathbb{R}^{d-1} \to \mathbb{R})$, then u solves

$$\frac{\partial_t u(t,x)}{s(1-s)\sqrt{1+|\nabla u|^2}} = \int_{\mathbb{R}^{d-1}} \frac{u(x+h)-u(x)}{|h|^{d+s}} F\left(\frac{u(x+h)-u(x)}{|h|}\right) dh, \tag{5}$$

where $F(L) = \frac{1}{L}\int_{-L}^{L} \frac{1}{(1+z^2)^{(d+s)/2}} dz$. Thus

$$F(||\nabla_x u||_{L^\infty}) \leq F\left(\frac{u(t,x+h)-u(t,x)}{|h|}\right) \leq 2, \tag{6}$$

so this is a nonlinear parabolic equation of order $1+s$, with the ellipticity constant depending on the Lipschitz constant of u.

Background

In the nongraphical case, fractional mean curvature flow is not defined for all time. For convex sets, $E_t \to \emptyset$ in finite time.

$$\text{Ex: } E_0 = B(0, R), \quad \Rightarrow \quad E_t = B(0, (R^{1+s} - c_{d,s}t)^{\frac{1}{1+s}}). \tag{7}$$

Background

In the nongraphical case, fractional mean curvature flow is not defined for all time. For convex sets, $E_t \to \emptyset$ in finite time.

$$\text{Ex: } E_0 = B(0,R), \quad \Rightarrow \quad E_t = B(0,(R^{1+s} - c_{d,s}t)^{\frac{1}{1+s}}). \tag{7}$$

In the nonconvex case, smooth initial data can develop singularities in finite time, after which the classical equation for the flow no longer makes sense. But we can define weak, viscosity solutions via the levelset method which will exist for all time.

Viscosity Solutions

Let (E_0^-, Γ_0, E_0^+) be a triple such that $E_0^\pm$ are open sets, Γ_0 is closed, and all are mutually disjoint with $E_0^- \cup \Gamma_0 \cup E_0^+ = \mathbb{R}^d$. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a globally Lipschitz function with

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \tag{8}$$

Viscosity Solutions

Let (E_0^-, Γ_0, E_0^+) be a triple such that $E_0^\pm$ are open sets, Γ_0 is closed, and all are mutually disjoint with $E_0^- \cup \Gamma_0 \cup E_0^+ = \mathbb{R}^d$. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a globally Lipschitz function with

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \tag{8}$$

By (Imbert, 2009), there exists a unique viscosity solution $U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ to the equation

$$\partial_t U(t, X) = -H_s(X, \{U(t, \cdot) \geq U(t, X)\})|\nabla_X U(t, X)|, \tag{9}$$

with $U(0, X) = U_0(X)$.

Viscosity Solutions

Let (E_0^-, Γ_0, E_0^+) be a triple such that $E_0^\pm$ are open sets, Γ_0 is closed, and all are mutually disjoint with $E_0^- \cup \Gamma_0 \cup E_0^+ = \mathbb{R}^d$. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a globally Lipschitz function with

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \qquad (8)$$

By (Imbert, 2009), there exists a unique viscosity solution $U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ to the equation

$$\partial_t U(t, X) = -H_s(X, \{U(t, \cdot) \geq U(t, X)\})|\nabla_X U(t, X)|, \qquad (9)$$

with $U(0, X) = U_0(X)$.
If we define the triple (E_t^-, Γ_t, E_t^+) by

$$E_t^- = \{U(t, \cdot) < 0\}, \Gamma_t = \{U(t, \cdot) = 0\}, E_t^+ = \{U(t, \cdot) > 0\}, \qquad (10)$$

then the triple (E_t^-, Γ_t, E_t^+) is unique and independent of the choice of U_0. This is the viscosity solution.

Viscosity Solutions

If $t \to E_t$ is a smooth flow, then $(E_t, \partial E_t, \mathbb{R}^d \setminus \overline{E_t})$ is the unique viscosity solution to the initial data $(E_0, \partial E_0, \mathbb{R}^d \setminus \overline{E_0})$.

Viscosity Solutions

If $t \to E_t$ is a smooth flow, then $(E_t, \partial E_t, \mathbb{R}^d \setminus \overline{E_t})$ is the unique viscosity solution to the initial data $(E_0, \partial E_0, \mathbb{R}^d \setminus \overline{E_0})$.

But if the flow develops singularities, then it's possible the for the viscosity solution to fatten. I.e., Γ_t can have a nonempty interior, in which case $\Gamma_t \neq \partial E_t^{\pm}$.

Viscosity Solutions

If $t \to E_t$ is a smooth flow, then $(E_t, \partial E_t, \mathbb{R}^d \setminus \overline{E_t})$ is the unique viscosity solution to the initial data $(E_0, \partial E_0, \mathbb{R}^d \setminus \overline{E_0})$.

But if the flow develops singularities, then it's possible the for the viscosity solution to fatten. I.e., Γ_t can have a nonempty interior, in which case $\Gamma_t \neq \partial E_t^{\pm}$.

Fattening represents the lack of uniqueness of the smooth flow $t \to E_t$, and a sensitivity to the way that we perturb our initial data.

Previous results for elliptic case

- (Caffarelli, Roquejoffre, Savin 2010) Generalizes classical results to nonlocal case. Density estimate, monotonicity formula, improvement of flatness, etc. Set of singular points has codimension at least 2 for any $s \in (0, 1)$.
- (Caffarelli, Valdinoci 2013) Set of singular points has codimension at least 8 for s sufficiently close to 1.
- (Figalli, Valdinoci 2017) Nonlocal minimal surfaces are smooth whenever they are locally Lipschitz, and Bernstein's theorem holds.
- (Cinti, Serra, Valdinoci 2018) BV estimate for stable nonlocal minimal surfaces, with quantitative flatness results for dimensions 2,3.
- (Davila, del Pino, Wei 2018) There exist stable s-minimal cones in $\mathbb{R}^7$ for small s.

Previous results for parabolic case

- (Imbert 2009/ Chambolle, Morini, Ponsiglione 2015) There exist unique, viscosity solutions for all time obeying the comparison principle.
- (Cinti, Sinestrari, Valdinoci 2018) There exists simply connected curves in the plane which can develop singularities in contrast to Grayson's theorem for local mean curvature flow.
- (Cesaroni, Dipierro, Novaga, Valdinoci 2018) Smooth strictly star-convex sets have unique flows but they can fatten if the initial data has a Lipschitz singularity. Number of examples of fattening/nonfattening behavior.

Our problem

From the elliptic case, we know that "flat implies smooth." So we should expect that a version of that is true in the parabolic case,

Our problem

From the elliptic case, we know that "flat implies smooth." So we should expect that a version of that is true in the parabolic case,

Claim

Let (E_0^-, Γ_0, E_0^+) *be a triple with* $\{x_d \leq 0\} \subseteq E_0^-, \{x_d \leq 1\} \subseteq E_0^+$*, and* (E_t^-, Γ_t, E_t^+) *be the viscosity solution. Then after flowing under s-fractional mean curvature flow for some finite, universal time* $T_{d,s}$*,* $\partial E_t^\pm$ *will be 1-Lipschitz graphs.*

Our problem

From the elliptic case, we know that "flat implies smooth." So we should expect that a version of that is true in the parabolic case,

Claim

Let (E_0^-, Γ_0, E_0^+) be a triple with $\{x_d \leq 0\} \subseteq E_0^-, \{x_d \leq 1\} \subseteq E_0^+$, and (E_t^-, Γ_t, E_t^+) be the viscosity solution. Then after flowing under s-fractional mean curvature flow for some finite, universal time $T_{d,s}$, $\partial E_t^{\pm}$ will be 1-Lipschitz graphs.

This is preliminary work, but we'll sketch the case when the flow $t \to E_t$ is smooth. The proof of the claim for the viscosity solutions should be posted on the arXiv in the next month.

Why this is interesting

The proof is inherently nonlocal in nature, so distinct from any previous proof in local mean curvature theory.
In fact, the claim is false for typical mean curvature, as the set

$$E_0 = \{x_d \leq 0\} \cup \{1/2 \leq x_d \leq 1\},$$

is fixed under mean curvature flow.

Why this is interesting

The proof is inherently nonlocal in nature, so distinct from any previous proof in local mean curvature theory.
In fact, the claim is false for typical mean curvature, as the set

$$E_0 = \{x_d \leq 0\} \cup \{1/2 \leq x_d \leq 1\},$$

is fixed under mean curvature flow.
This is a reflection of the fact that ∂E_0 is a union of disjoint planes, so the obstruction to regularity is multiplicity. But multiplicity is not a possibility for fractional mean curvature flow because of its nonlocal nature, and direct calculation shows that $E_0 \to \{x_d \leq 1/2\}$ in finite time for any $s \in (0, 1)$.

Inspiration

Our proof is inspired by Kiselev's proof of eventual Holder regulariztion for supercritical Burger's equation.
For any $\alpha \in [0, 1]$, we define the α-Burger's equation to be

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) + (-\Delta)^\alpha u(t,x) = 0. \tag{11}$$

For any $\alpha \geq 1/2$ (subcritical-critical case) its known that smooth initial data has a smooth solution, but for $0 \leq \alpha < 1/2$ smooth initial data can become discontinuous in finite time.

Inspiration

However, consider

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) + (-\Delta)^\alpha u(t,x) - \epsilon \Delta u(t,x) = 0, \quad \alpha \in (0,1/2), \epsilon > 0. \tag{12}$$

Inspiration

However, consider

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) + (-\Delta)^\alpha u(t,x) - \epsilon\Delta u(t,x) = 0, \\ \alpha \in (0,1/2), \epsilon > 0. \tag{12}$$

Kiselev showed in 2011 that solutions becomes Holder continuous in finite time with an estimate independent of ϵ. For the proof, he showed that the equation propagated a family of moduli of continuity

$$\omega(t,r) \approx \delta(t) + Cr^\beta,$$

where $\delta(0) > 2||u_0||_{L^\infty}$ and $\delta(T_{\alpha,\beta}) = 0$. Thus the moduli of continuity gives no new information at time $t = 0$, but forces the solution to become β-Holder continuous at time $t = T_{\alpha,\beta}$.

Inspiration

However, consider

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) + (-\Delta)^\alpha u(t,x) - \epsilon \Delta u(t,x) = 0, \quad \alpha \in (0,1/2), \epsilon > 0. \tag{12}$$

Kiselev showed in 2011 that solutions becomes Holder continuous in finite time with an estimate independent of ϵ. For the proof, he showed that the equation propagated a family of moduli of continuity

$$\omega(t,r) \approx \delta(t) + Cr^\beta,$$

where $\delta(0) > 2||u_0||_{L^\infty}$ and $\delta(T_{\alpha,\beta}) = 0$. Thus the moduli of continuity gives no new information at time $t = 0$, but forces the solution to become β-Holder continuous at time $t = T_{\alpha,\beta}$. Since these moduli don't converge to 0, its possible to apply these more general moduli to boundaries of sets which can fold over themselves. But then at time $t = T$, satisfying the modulus forces the boundary to be graphical.

Moduli of continuity for sets

Fix a smooth set $E_0 \subseteq \mathbb{R}^d$ with

$$\{x_d \leq 0\} \subseteq E_0 \subseteq \{x_d \leq 1\}.$$

Then by the comparison principle, the same is true for E_t for all times t.

Define the upper and lower boundaries of E_t by

$$\overline{u}(t,x) = \sup\{z | (x,z) \in \partial E_t\}, \qquad \underline{u}(t,x) = \inf\{z | (x,z) \in \partial E_t\}. \tag{13}$$

If $\omega : [0,\infty) \to [0,\infty)$, then we say that E_t has modulus ω if

$$\overline{u}(t,x) - \underline{u}(t,y) \leq \omega(|x-y|), \qquad \forall x, y \in \mathbb{R}^{d-1}. \tag{14}$$

Equivalently, $E_t \subseteq E_t + (h, \omega(|h|))$ for all $h \in \mathbb{R}^{d-1}$.

Moduli of continuity for sets

If E_0 has modulus of continuity ω, then by comparison principle

$$E_0 \subseteq E_0 + (h, \omega(|h|)) \quad \Rightarrow E_t \subseteq E_t + (h, \omega(|h|)).$$

Thus E_t has modulus of continuity ω as well for all time t.

Moduli of continuity for sets

If E_0 has modulus of continuity ω, then by comparison principle

$$E_0 \subseteq E_0 + (h, \omega(|h|)) \quad \Rightarrow E_t \subseteq E_t + (h, \omega(|h|)).$$

Thus E_t has modulus of continuity ω as well for all time t.
Rather than just propagation, we want to show an improvement in our modulus. We want a time dependent family

$$\omega(t, r) : [0, \infty) \times [0, \infty) \to [0, \infty),$$

such that

1. $\omega(0, r) > 1$,
2. $\omega(t, \cdot)$ is $C^{1,1}$, with $\omega(t, 0) > 0$, $0 \leq \partial_r \omega(t, r) \leq 1$, and $\partial_r \omega(t, 0) = 0$ for all $t \in [0, T)$,
3. $\omega(T, \cdot)$ is 1-Lipschitz with $\omega(T, 0) = 0$,

Then proving E_t has modulus $\omega(t, \cdot)$ for all $t \in [0, T]$ would prove the claim.

Breakthrough Argument

Assume that we have a smooth flow $t \to E_t$, and that $\{x_d \leq 0\} \subseteq E_0 \subseteq \{x_d \leq 1\}$ with

$$\lim_{|x|\to\infty} \overline{u}(0,x) = \lim_{|x|\to\infty} \underline{u}(0,x) = 1/2. \tag{15}$$

By comparison principle this will be true for E_t for all t. Let $\omega(t,r)$ be a time dependent family of moduli of continuity satisfying out assumptions. Then E_0 has modulus $\omega(0,\cdot)$, and by continuity it will have modulus $\omega(t,\cdot)$ for small t.

Breakthrough Argument

Assume that we have a smooth flow $t \to E_t$, and that $\{x_d \leq 0\} \subseteq E_0 \subseteq \{x_d \leq 1\}$ with

$$\lim_{|x|\to\infty} \overline{u}(0,x) = \lim_{|x|\to\infty} \underline{u}(0,x) = 1/2. \qquad (15)$$

By comparison principle this will be true for E_t for all t. Let $\omega(t,r)$ be a time dependent family of moduli of continuity satisfying out assumptions. Then E_0 has modulus $\omega(0,\cdot)$, and by continuity it will have modulus $\omega(t,\cdot)$ for small t.

Suppose E_t loses this modulus after time $t^* \in (0,T)$. Then necessarily

$$\overline{u}(t^*,x) - \underline{u}(t^*,y) \leq \omega(t^*,|x-y|), \qquad \forall x,y \in \mathbb{R}^{d-1} \qquad (16)$$

Breakthrough Argument

Assume that we have a smooth flow $t \to E_t$, and that $\{x_d \leq 0\} \subseteq E_0 \subseteq \{x_d \leq 1\}$ with

$$\lim_{|x| \to \infty} \overline{u}(0, x) = \lim_{|x| \to \infty} \underline{u}(0, x) = 1/2. \tag{15}$$

By comparison principle this will be true for E_t for all t. Let $\omega(t, r)$ be a time dependent family of moduli of continuity satisfying out assumptions. Then E_0 has modulus $\omega(0, \cdot)$, and by continuity it will have modulus $\omega(t, \cdot)$ for small t.

Suppose E_t loses this modulus after time $t^* \in (0, T)$. Then necessarily

$$\overline{u}(t^*, x) - \underline{u}(t^*, y) \leq \omega(t^*, |x - y|), \qquad \forall x, y \in \mathbb{R}^{d-1} \tag{16}$$

If the inequality was strict for all x, y, then decay assumption on ∂E_t and $\omega(t^*, 0) > 0 \ \Rightarrow$

$$\overline{u}(t^* + \epsilon, x) - \underline{u}(t^* + \epsilon, y) < \omega(t^* + \epsilon, |x - y|), \tag{17}$$

for some $\epsilon > 0$ sufficiently small.

Breakthrough Argument

Thus if E_t loses modulus $\omega(t, \cdot)$ after time t^*, $\Rightarrow$ $\exists x, y \in \mathbb{R}^{d-1}$ s.t.

$$\overline{u}(t^*, x) - \underline{u}(t^*, y) = \omega(t^*, |x - y|). \tag{18}$$

Thus we just need to find a modulus ω s.t. this $\Rightarrow$

$$\frac{d}{dt}\left(\overline{u}(t, x) - \underline{u}(t, y)\right)\Big|_{t=t^*} < \partial_t \omega(t^*, |x - y|), \tag{19}$$

contradicting the fact that E_t had modulus $\omega(t, \cdot)$ for time $t < t^*$.

Modulus estimates

Fix time at $t = t^*$. Assume that

$$\begin{cases} \overline{u}(x) - \underline{u}(y) \leq \omega(|x-y|), \\ \overline{u}(\xi/2) - \underline{u}(-\xi/2) = \omega(|\xi|), \end{cases} \tag{20}$$

for some modulus ω and $\xi \in \mathbb{R}^{d-1}$. What can we say?

Modulus estimates

Fix time at $t = t^*$. Assume that

$$\begin{cases} \overline{u}(x) - \underline{u}(y) \leq \omega(|x-y|), \\ \overline{u}(\xi/2) - \underline{u}(-\xi/2) = \omega(|\xi|), \end{cases} \tag{20}$$

for some modulus ω and $\xi \in \mathbb{R}^{d-1}$. What can we say?
$\overline{u}$ is touched from above by ω at $\xi/2$ and $\underline{u}$ from below by $-\omega$ at $-\xi/2$, so $\Rightarrow$

$$\nabla\overline{u}(\xi/2) = \nabla\underline{u}(-\xi/2) = \omega'(|\xi|)\hat{\xi}. \tag{21}$$

Note that as $\omega'(0) = 0$ by assumption, the statement is still true when $\xi = 0$.

Modulus estimates

The normal vector to ∂E at the point $(\xi/2, \overline{u}(\xi/2))$ is $\frac{(-\omega'(|\xi|)\hat{\xi}, 1)}{\sqrt{1+\omega'(|\xi|)^2}}$. Thus

$$\partial_t \overline{u}(\xi/2) = s(1-s)\sqrt{1+\omega'(|\xi|)^2} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}(\frac{\xi}{2}+h, \overline{u}(\frac{\xi}{2})+z)}{(|h|^2+z^2)^{(d+s)/2}} dhdz \tag{22}$$

where $\mathbb{1}_E^{\pm} = \mathbb{1}_E - \mathbb{1}_{CE}$.

Modulus estimates

$$\frac{\partial_t(\overline{u}(\xi/2) - \underline{u}(-\xi/2))}{s(1-s)\sqrt{1+\omega'(|\xi|)^2}} =$$

$$\int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}(\frac{\xi}{2}+h, \overline{u}(\frac{\xi}{2})+z) - \mathbb{1}_E^{\pm}(\frac{-\xi}{2}+h, \underline{u}(\frac{-\xi}{2})+z)}{(|h|^2+z^2)^{(d+s)/2}} dhdz \tag{23}$$

Because E has the modulus ω and $\overline{u}(\xi/2) - \underline{u}(\xi/2) = \omega(|\xi|)$, you get immediately that

$$\mathbb{1}_E^{\pm}(\frac{\xi}{2}+h, \overline{u}(\frac{\xi}{2})+z) - \mathbb{1}_E^{\pm}(\frac{-\xi}{2}+h, \underline{u}(\frac{-\xi}{2})+z) \leq 0, \tag{24}$$

for all $(h, z) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Now we just need to quantify how often it is negative in terms of ω.

Modulus estimate

After some mostly straightforward bounds and rearranging integrals, get

$$\frac{\partial_t(\overline{u}(\xi/2) - \underline{u}(-\xi/2))}{s(1-s)\sqrt{1+\omega'(|\xi|)^2}} \leq -\int_{\mathbb{R}^{d-1}} (\overline{u}(h) - \underline{u}(h)) \min\{K(h \pm \frac{\xi}{2})\} dh$$
$$- \int_{\mathbb{R}^{d-1}} (\omega(|\xi|) - \overline{u}(\frac{\xi}{2} + h) + \underline{u}(\frac{-\xi}{2} + h)) K(h) dh, \tag{25}$$

where

$$K(h) = \frac{1}{(|h|^2 + \omega(|h|)^2)^{(d+s)/2}}. \tag{26}$$

Modulus estimate

Rearranging terms some and using that E has modulus ω, can get estimate purely in terms of ω

$$\lesssim \int_0^{|\xi|/2} \frac{\omega(|\xi|+2\eta)+\omega(|\xi|-2\eta)-2\omega(|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}}\,d\eta \\ + \int_{|\xi|/2}^{\infty} \frac{\omega(2\eta+|\xi|)-\omega(2\eta-|\xi|)-2\omega(|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}}\,d\eta. \tag{27}$$

Modulus estimate

Rearranging terms some and using that E has modulus ω, can get estimate purely in terms of ω

$$\begin{aligned} &\lesssim \int_0^{|\xi|/2} \frac{\omega(|\xi|+2\eta)+\omega(|\xi|-2\eta)-2\omega(|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}}\,d\eta \\ &\qquad + \int_{|\xi|/2}^{\infty} \frac{\omega(2\eta+|\xi|)-\omega(2\eta-|\xi|)-2\omega(|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}}\,d\eta. \end{aligned} \tag{27}$$

With estimates out of the way, our goal is now to actually choose our family of moduli $\omega(t,r)$ so that

1. $\omega(0,r) > 1$,
2. $\omega(t,\cdot)$ is $C^{1,1}$, with $\omega(t,0) > 0$, $0 \le \partial_r\omega(t,r) \le 1$, and $\partial_r\omega(t,0) = 0$ for all $t \in [0,T)$,
3. $\omega(T,\cdot)$ is 1-Lipschitz with $\omega(T,0) = 0$,
4. $(27)(t,|\xi|) < \partial_t\omega(t,|\xi|)$ for all $t \in (0,T)$

Our choice of modulus

Now we need to find a choice of $\omega(t,r)$. First guess might be $\omega(t,r) = \delta(t) + r$. But, generally you need to have some strict concavity in r in order to get estimates with this method.

Our choice of modulus

Now we need to find a choice of $\omega(t, r)$. First guess might be $\omega(t, r) = \delta(t) + r$. But, generally you need to have some strict concavity in r in order to get estimates with this method. Second guess might be $\omega(t, r) = \delta(t) + r - \frac{r^{1+s}}{2^{1+s}}$, where exponent $1 + s$ was chosen for scaling reasons. But we need the modulus $\omega(t, \cdot)$ to be $C^{1,1}$ in $\mathbb{R}^{d-1}$

Our choice of modulus

Now we need to find a choice of $\omega(t,r)$. First guess might be $\omega(t,r) = \delta(t) + r$. But, generally you need to have some strict concavity in r in order to get estimates with this method.
Second guess might be $\omega(t,r) = \delta(t) + r - \frac{r^{1+s}}{2^{1+s}}$, where exponent $1+s$ was chosen for scaling reasons. But we need the modulus $\omega(t,\cdot)$ to be $C^{1,1}$ in $\mathbb{R}^{d-1}$
Cutting it off for small r and replace with a parabola, our final choice ends up being

$$\omega(t,r) = \delta(t) + \begin{cases} 1, & 2 \le r \\ r - \frac{r^{1+s}}{2^{1+s}}, & c\delta^2 \le r \le 2, \\ A(\delta)r^2 + B(\delta), & 0 \le r \le c\delta^2 \end{cases} \tag{28}$$

where $c << 1$ and $A(\delta), B(\delta)$ are chosen so that $\omega(t,\cdot)$ is C^1.

Our choice of modulus

With this choice for $\omega(t,r)$, its straightforward to show that

$$\int_0^{|\xi|/2} \frac{\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}}d\eta + \int_{|\xi|/2}^{\infty} \frac{\omega(t,2\eta+|\xi|)-\omega(t,2\eta-|\xi|)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}}d\eta \lesssim -1. \tag{29}$$

Taking $\delta(t) = 1 - \frac{t}{T}$ for T sufficiently large, we then have that

$$\frac{d}{dt}\left(\overline{u}(t,x) - \underline{u}(t,y)\right)\Big|_{t=t^*} < \partial_t\omega(t^*,|x-y|), \tag{30}$$

completing the proof.

Extending the proof to viscosity solutions

Let (E_0^-, Γ_0, E_0^+) be such that $\{x_d \leq 0\} \subseteq E_0^-, \{x_d \leq 1\} \subseteq E_0^+$ and (E_t^-, Γ_t, E_t^+) be the viscosity solution.

Extending the proof to viscosity solutions

Let (E_0^-, Γ_0, E_0^+) be such that $\{x_d \leq 0\} \subseteq E_0^-, \{x_d \leq 1\} \subseteq E_0^+$ and (E_t^-, Γ_t, E_t^+) be the viscosity solution.
If we knew a priori that $\mathcal{L}^d(\Gamma_t) = 0$ for all times t, then we can tweak the previous argument to work for a viscosity solution. The indicator functions $\mathbb{1}_{E_t^- \cup \Gamma_t}(X)$, and $\mathbb{1}_{E_t^-}(X)$ are respectively viscosity sub and super solutions to

$$\partial_t U(t,X) = -H_s(X, \{U(t,\cdot) \geq U(t,X)\})|\nabla_X U(t,X)|. \tag{31}$$

The difference $\mathbb{1}_{E_t^- \cup \Gamma_t}(X) - \mathbb{1}_{E_t^-}(Y)$ will then be a subsolution to

Extending the proof to viscosity solutions

Let (E_0^-, Γ_0, E_0^+) be such that $\{x_d \leq 0\} \subseteq E_0^-, \{x_d \leq 1\} \subseteq E_0^+$ and (E_t^-, Γ_t, E_t^+) be the viscosity solution.
If we knew a priori that $\mathcal{L}^d(\Gamma_t) = 0$ for all times t, then we can tweak the previous argument to work for a viscosity solution. The indicator functions $\mathbb{1}_{E_t^- \cup \Gamma_t}(X)$, and $\mathbb{1}_{E_t^-}(X)$ are respectively viscosity sub and super solutions to

$$\partial_t U(t, X) = -H_s(X, \{U(t, \cdot) \geq U(t, X)\})|\nabla_X U(t, X)|. \tag{31}$$

The difference $\mathbb{1}_{E_t^- \cup \Gamma_t}(X) - \mathbb{1}_{E_t^-}(Y)$ will then be a subsolution to

$$\begin{aligned} \partial_t F(t, X, Y) = -H_s(X, \{F(t, \cdot, Y) \geq F(t, X, Y)\})|\nabla_X F| \\ + H_s(Y, \{F(t, X, \cdot) > F(t, X, Y)\})|\nabla_Y F|. \end{aligned} \tag{32}$$

Extending the proof to viscosity solutions

Let (E_0^-, Γ_0, E_0^+) be such that $\{x_d \leq 0\} \subseteq E_0^-, \{x_d \leq 1\} \subseteq E_0^+$ and (E_t^-, Γ_t, E_t^+) be the viscosity solution.
If we knew a priori that $\mathcal{L}^d(\Gamma_t) = 0$ for all times t, then we can tweak the previous argument to work for a viscosity solution. The indicator functions $\mathbb{1}_{E_t^- \cup \Gamma_t}(X)$, and $\mathbb{1}_{E_t^-}(X)$ are respectively viscosity sub and super solutions to

$$\partial_t U(t, X) = -H_s(X, \{U(t, \cdot) \geq U(t, X)\})|\nabla_X U(t, X)|. \tag{31}$$

The difference $\mathbb{1}_{E_t^- \cup \Gamma_t}(X) - \mathbb{1}_{E_t^-}(Y)$ will then be a subsolution to

$$\begin{aligned} \partial_t F(t, X, Y) = &-H_s(X, \{F(t, \cdot, Y) \geq F(t, X, Y)\})|\nabla_X F| \\ &+ H_s(Y, \{F(t, X, \cdot) > F(t, X, Y)\})|\nabla_Y F|. \end{aligned} \tag{32}$$

The test function $\Omega(t, X, Y) = \omega(t, |x - y|) + (1 - x_d)_+ + (y_d)_+$ then encodes whether the sets $E_t^\pm$ have modulus $\omega(t, \cdot)$. All our previous calculations go through at crossing points, so long as $\mathcal{L}^d(\Gamma_t) = 0$.

Unfortunately Γ_t can fatten, so we can't guarantee it won't have positive measure. So we adjust. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function such that

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \tag{33}$$

and let $U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ to the unique viscosity solution with initial data U_0.

Unfortunately Γ_t can fatten, so we can't guarantee it won't have positive measure. So we adjust. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function such that

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \quad (33)$$

and let $U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ to the unique viscosity solution with initial data U_0.
For any value $\gamma \in \mathbb{R}$, let

$$\Gamma_t^\gamma = \{U(t, \cdot) = \gamma\}. \quad (34)$$

In particular, $\Gamma_t^0 = \Gamma_t$. Then for all but countably many γ,

$$\mathcal{L}^{d+1}(U^{-1}(\gamma)) = 0 \quad \Rightarrow \quad \mathcal{L}^d(\Gamma_t^\gamma) = 0 \quad \text{for a.e. } t. \quad (35)$$

Unfortunately Γ_t can fatten, so we can't guarantee it won't have positive measure. So we adjust. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function such that

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \tag{33}$$

and let $U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ to the unique viscosity solution with initial data U_0.

For any value $\gamma \in \mathbb{R}$, let

$$\Gamma_t^\gamma = \{U(t, \cdot) = \gamma\}. \tag{34}$$

In particular, $\Gamma_t^0 = \Gamma_t$. Then for all but countably many γ,

$$\mathcal{L}^{d+1}(U^{-1}(\gamma)) = 0 \quad \Rightarrow \quad \mathcal{L}^d(\Gamma_t^\gamma) = 0 \quad \text{for a.e. } t. \tag{35}$$

Extend the smooth argument to work for any such γ, getting that Γ_T^γ is a 1-Lipschitz graph for almost any γ.

Unfortunately Γ_t can fatten, so we can't guarantee it won't have positive measure. So we adjust. Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function such that

$$E_0^- = \{U_0 < 0\}, \Gamma_0 = \{U_0 = 0\}, E_0^+ = \{U_0 > 0\}. \tag{33}$$

and let $U : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ to the unique viscosity solution with initial data U_0.

For any value $\gamma \in \mathbb{R}$, let

$$\Gamma_t^\gamma = \{U(t, \cdot) = \gamma\}. \tag{34}$$

In particular, $\Gamma_t^0 = \Gamma_t$. Then for all but countably many γ,

$$\mathcal{L}^{d+1}(U^{-1}(\gamma)) = 0 \quad \Rightarrow \quad \mathcal{L}^d(\Gamma_t^\gamma) = 0 \quad \text{for a.e. } t. \tag{35}$$

Extend the smooth argument to work for any such γ, getting that Γ_T^γ is a 1-Lipschitz graph for almost any γ.

Taking $0 > \gamma_n^- \to 0$ and $0 < \gamma_n^+ \to 0$, you then get that

$$\Gamma_T^{\gamma_n^\pm} \to \partial E_T^\pm, \tag{36}$$

so they are 1-Lipschitz graphs as well.

Thank you