Emergence of exponentially weighted L^p-norms and Sobolev regularity for the Boltzmann equation

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In this talk we consider:

- **Homogeneous equation without cutoff with full range of angular singularity.**
- **The case of Maxwell and hard potentials (the later is singular in velocity growth).**
- **L^p theory (including the case $p=\infty$) with exponential weights based in the L^1 and L^2 theories.**
- **Sobolev regularity with exponential weights.**

The Boltzmann model

$$\partial_t f(t,v) = Q(f,f)(t,v)\,, \qquad (t,v) \in (0,\infty) \times \mathbb{R}^d$$

The collision operator

Maxwell and Hard $\gamma \geqq 0$

$$Q(g,f)(v) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \big(g(v'_*)f(v') - g(v_*)f(v)\big)|u|^\gamma b(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*$$

The fractional diffusion: 0<s<1

$$\sin^{d-2}\theta\, b(\cos\theta) \approx b_0/\theta^{1+2s}\,, \quad \text{when} \quad \theta \approx 0$$

Coercivity estimate:
Importance of conservation laws and entropy

L^p estimate for the Heat equation

$$\partial_t \| f \|_p^p + \frac{4}{p'} \| \nabla f^{p/2} \|_2^2 = 0 \,, \qquad p > 1 \,.$$

Needs integration by parts and explicit estimation of the Dirichlet product

$$\langle \nabla f, \nabla f^{p-1} \rangle_{L^2}$$

For the collision operator

In the space

$$\mathcal{U}(D_0, E_0) = \Big\{ g \ \text{ measurable} : g \geq 0\,, \ \int_{\mathbb{R}^d} g \, \mathrm{d}v \geq D_0\,, \ \int_{\mathbb{R}^d} g\,(1+|v|^2+\ln g)\,\mathrm{d}v \leq E_0 \Big\}$$

we have that

$$-\Big(Q(g,f)\,,\, f\Big)_{L^2} \geq c_0 \| \langle v \rangle^{\gamma/2} f \|^2_{H^s} - C \| \langle v \rangle^{\gamma/2} f \|^2_{H^{(-\gamma/2)+}}$$

based on the fact that

$$\mathcal{C}_\gamma(g,\, f) = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(.)|v-v_*|^\gamma g_*(f'-f)^2 dv dv_* d\sigma,$$

$$\geq c_0' \| \langle v \rangle^{\gamma/2} f \|^2_{H^s} - C' \| f \|^2_{L^2_{\gamma/2}}$$

Important lemma for the Dirichlet product

Lemma 1. *For any* $\theta \geq 0$

$$\theta^{2/p'} - 1 \leq \frac{1}{p'}(\theta^2 - 1) - \frac{1}{\max\{p, p'\}}(\theta - 1)^2, \qquad p \in (1, \infty]. \tag{3}$$

In particular, in the limit $p \to 1$ *it follows that*

$$2 \log \theta \leq \left(\theta^2 - 1\right) - (\theta - 1)^2.$$

Note that equality is achieved in estimate (3) *for the case* $p = 2$.

Simple argument…

$$
\begin{aligned}
F(v)\Big(F(v')^{p-1} - F(v)^{p-1}\Big) &= F(v)^p\left(\Big(\frac{F(v')^{p/2}}{F(v)^{p/2}}\Big)^{2/p'} - 1\right) \\
&\le F(v)^p\left(\frac{1}{p'}\Big(\frac{F(v')^p}{F(v)^p} - 1\Big) - \frac{1}{\max\{p,p'\}}\Big(\frac{F(v')^{p/2}}{F(v)^{p/2}} - 1\Big)^2\right) \\
&= \frac{1}{p'}\Big(F(v')^p - F(v)^p\Big) - \frac{1}{\max\{p,p'\}}\Big(F(v')^{p/2} - F(v)^{p/2}\Big)^2
\end{aligned}
$$

Cancellation lemma **Coercivity estimate**

Energy estimate for emergence of L^p norms

Proposition 1. *Take* $g \in \mathcal{U}(D_0, E_0)$, *$F$ sufficiently smooth,* $\gamma \in [0,1]$, $s \in (0,1)$, *and* $p \in (1,\infty)$. *Then,*

$$\int_{\mathbb{R}^d} Q(g,F)(v)F^{p-1}(v)\,dv \le -\frac{c_g}{\max\{p,p'\}}\|\langle\cdot\rangle^{\gamma/2}F^{p/2}\|^2_{H^s} + \frac{C_g}{p'}\|\langle\cdot\rangle^{\gamma/2}F^{p/2}\|^2_2\,,$$

where the constants c_g *and* C_g *depend on* D_0, E_0, d, γ, *and* s.

Which leads to

$$\|f(t)\|_p \le C_p\|\langle\cdot\rangle^2 f_0\|_1\left(\frac{1}{t^{\frac{d}{2sp'}}}+1\right), \quad p \in (1,\infty)$$

Special case - L∞ case: A De Giorgi argument

$$F_K(v) := F(v) - K \text{ and } F_K^+(v) := F_K(v)1_{\{F_K \geq 0\}}$$

Compute

$$\begin{aligned}
&F(v)\Big(F_K(v')1_{\{F'_K \geq 0\}} - F_K(v)1_{\{F_K \geq 0\}}\Big) \\
&\quad = F_K(v)\Big(F_K(v')1_{\{F'_K \geq 0\}} - F_K(v)1_{\{F_K \geq 0\}}\Big) + K\Big(F_K(v')1_{\{F'_K \geq 0\}} - F_K(v)1_{\{F_K \geq 0\}}\Big) \\
&\quad = F_K(v)\Big(1_{\{F_K \geq 0\}} + 1_{\{F_K < 0\}}\Big)\Big(F_K(v')1_{\{F'_K \geq 0\}} - F_K(v)1_{\{F_K \geq 0\}}\Big) \\
&\qquad\qquad + K\Big(F_K(v')1_{\{F'_K \geq 0\}} - F_K(v)1_{\{F_K \geq 0\}}\Big).
\end{aligned}$$

Note that

$$F_K(v)1_{\{F_K < 0\}}\Big(F_K(v')1_{\{F'_K \geq 0\}} - F_K(v)1_{\{F_K \geq 0\}}\Big) = F_K(v)1_{\{F_K < 0\}}F_K(v')1_{\{F'_K \geq 0\}} \leq 0$$

Then,

$$
\begin{aligned}
F(v)\Big(F_K(v')1_{\{F'_K\geq 0\}} - F_K(v)1_{\{F_K\geq 0\}}\Big)
&\leq F_K(v)1_{\{F_K\geq 0\}}\Big(F_K(v')1_{\{F'_K\geq 0\}} - F_K(v)1_{\{F_K\geq 0\}}\Big) \\
&\qquad + K\Big(F_K(v')1_{\{F'_K\geq 0\}} - F_K(v)1_{\{F_K\geq 0\}}\Big) \\
&= \tfrac{1}{2}\Big(F_K^+(v')^2 - F_K^+(v)^2\Big) - \tfrac{1}{2}\Big(F_K^+(v') - F_K^+(v)\Big)^2 + K\Big(F_K^+(v') - F_K^+(v)\Big)
\end{aligned}
$$

L^2 remainder **Coercive part** **L^1 remainder**

This leads to

Proposition 2. *Take $g \in \mathcal{U}(D_0, E_0)$, F sufficiently smooth, $\gamma \in [0,1]$, and $s \in (0,1)$. Then,*

$$
\int_{\mathbb{R}^d} Q(g,F)(v)F_K^+(v)\,dv \leq -c_g\|\langle\cdot\rangle^{\gamma/2}F_K^+\|_{H^s}^2 + C_g\|\langle\cdot\rangle^{\gamma/2}F_K^+\|_2^2 + C_g\,K\,\|\langle\cdot\rangle^{\gamma}F_K^+\|_1\,,
$$

where the constants c_g and C_g depend on D_0, E_0, d, γ, and s.

Level set energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_K^+\|_2^2 + c_0\|\langle\cdot\rangle^{\gamma/2}f_K^+\|_{H^s}^2 \leq C_0\|\langle\cdot\rangle^{\gamma/2}f_K^+\|_2^2 + C_0\,K\,\|\langle\cdot\rangle^{\gamma}f_K^+\|_1$$

Energy functional

$$W_k := \tfrac{1}{2}\sup_{t\in[t_k,T]}\|f_k(t)\|_2^2 + c_0\int_{t_k}^{T}\|\langle\cdot\rangle^{\gamma/2}f_k(s)\|_{H^s}^2\mathrm{d}s\,, \qquad T > t_* > 0\,.$$

Sobolev embedding and the key observation

$$1_{\{f\geq K_k\}} \leq \Big(\frac{2^k}{K}f_{k-1}\Big)^{\alpha}, \qquad \forall\,\alpha\geq 0\,.$$

gives that

$$\tfrac{1}{2}W_k \leq 2^{\frac{2s}{d}}\frac{2^{\frac{d+4s}{d}k}}{c_0K^{\frac{4s}{d}}}\Big(\frac{1}{t_*}+2C_0\Big)W_{k-1}^{\frac{d+2s}{d}}$$

This proves the theorem

Theorem 2 (Generation/propagation of L^∞-norm)**.** *Let $\gamma \in [0,1]$ and $s \in (0,1)$, and assume $f_0 \in \mathcal{U}(D_0, E_0)$. Then,*

$$\|f(t)\|_\infty \le C \|\langle\cdot\rangle^2 f_0\|_1 \left(\frac{1}{t^{\frac{d}{2s}}} + 1 \right), \qquad t > 0, \tag{22}$$

where C depends on D_0, E_0, d, γ and s. Furthermore, if additionally $f_0 \in L^\infty(\mathbb{R}^d)$, then,

$$\sup_{t \ge 0} \|f(t)\|_\infty \le \max\left\{ 2\|f_0\|_\infty, C\|\langle\cdot\rangle^2 f_0\|_1 \right\}, \tag{23}$$

where C depends also on $\|f_0\|_2$.

We work now with polynomial and exponential weights

$$w(v) = \langle v \rangle^k \ \ or \ w(v) = e^{r \min\{t,1\} \langle v \rangle^\gamma}$$

$$w(v) = e^{r\langle v \rangle^\alpha}, \text{ with } \alpha \in (0,1]$$

Adding weights to previous estimates

$$F(v) = f(v)\, w(v),\; G(v) = g(v)\, w(v)$$

Compute

$$\begin{aligned}
f(v)\Big(F(v')^{p-1}w(v') - F(v)^{p-1}w(v)\Big) \\
&= F(v)\Big(F(v')^{p-1} - F(v)^{p-1}\Big) + f(v)F(v')^{p-1}\big(w(v') - w(v)\big) \\
&= F(v)\Big(F(v')^{p-1} - F(v)^{p-1}\Big) + f(v)\Big(F(v')^{p-1} - F(v)^{p-1}\Big)\big(w(v') - w(v)\big) \\
&\quad + f(v)F(v)^{p-1}\big(w(v') - w(v)\big)\,.
\end{aligned}$$

The first is the leading term, already computed. Second and third terms are remainder terms. The estimation of these terms rely in several classical inequalities such as the cancellation lemma and

$$\big|\varphi(v') - \varphi(v)\big| \le \sqrt{2} \sup_{|x|\le\sqrt{|v|^2+|v_*|^2}} \big|\partial\varphi(x)\big| \times |u| \sin(\theta)\,,$$

$$\Big|\int_{\mathbb{S}^{d-2}} \varphi(v') - \varphi(v)\, dw\Big| \le \big|\mathbb{S}^{d-2}\big| \sup_{|x|\le\sqrt{|v|^2+|v_*|^2}} \big|\partial^2\varphi(x)\big| \times |u|^2 \sin^2(\theta)\,.$$

The final estimate is given by

Lemma 2. *Assume that* $g \in \mathcal{U}(D_0, E_0)$, $\gamma \in [0,1]$, *and set* $F(v) = f(v)\,w(v)$ *and* $G(v) = g(v)\,w(v)$, *for polynomial or exponential weight* w. *Then, for any* $\delta > 0$

$$\int_{\mathbb{R}^d} Q(g,f)(v)F(v)^{p-1}w(v)\,dv \le -\Big(\frac{c_0}{\max\{p,p'\}} - \delta\|\langle\cdot\rangle^{2\gamma}G\|_1\Big)\|\langle\cdot\rangle^{\gamma/2}F^{p/2}\|^2_{H^s} + \Big(\frac{C_0}{p'} + (C+\delta)\|\langle\cdot\rangle^{3\gamma}G\|_1\Big)\|\langle\cdot\rangle^{3\gamma/p}F\|^p_p + \frac{C}{\delta}\|\langle\cdot\rangle^{4\gamma+2s}G\|_1\|\langle\cdot\rangle^{\frac{3\gamma+2s}{p}}F\|^p_p\,. \tag{29}$$

The constants C_0 *and* c_0 *depend on* D_0, E_0, d, γ, s. *The constant* C *depends also on the corresponding weight parameter* $k \ge 0$ *or* $r > 0$.

Generation of moments and Lemma 2 lead to the estimate

$$\|\langle\cdot\rangle^k f(t)\|_p \le C\left(\frac{1}{t^{\frac{ck}{s\gamma}}} + 1\right), \qquad k \ge 0\,,$$

$$\|e^{r\min\{t,1\}\langle\cdot\rangle^\gamma} f(t)\|_p \le C\left(\frac{1}{t^{\frac{c}{s\gamma}}} + 1\right), \qquad p \in (1,\infty)\,, \quad r \in \big(0, r_0\big)\,.$$

- **Rates include emergence of moments and regularity.**
- **Constants depend only mass, energy and entropy**

The L∞ case with exponential weights

Notation

$$F_K(v) = \big(f(v)w(v) - K\big)$$

$$F_K^+ = F_K(v)1_{\{F_K \geq 0\}}$$

$$F(v) := f(v)w(v)$$

Compute

$$f(v)\Big(F_K^+(v')w(v') - F_K^+(v)w(v)\Big) \leq F_K^+(v)\Big(F_K^+(v') - F_K^+(v)\Big) + K\Big(F_K^+(v') - F_K^+(v)\Big) + f(v)F_K^+(v')\big(w(v') - w(v)\big)$$

Only one extra term

Again, one is lead to

Proposition 3. *Take* $\gamma \in [0,1]$ *and* $s \in (0,1)$, *and assume* $f \in \mathcal{U}(D_0, E_0)$ *is sufficiently smooth and decaying. Consider also, for any* $K \geq 0$, *the level function* $F_K^+ = (fw - K)1_{\{fw-K\geq 0\}}$. *Then,*

$$\int_{\mathbb{R}^d} Q(f,f)(v)F_K^+(v)w(v)\,dv \leq -\frac{c_0}{2}\|\langle\cdot\rangle^{\gamma/2}F_K^+\|_{H^s}^2$$
$$+\frac{C}{(1-s)}\|\langle\cdot\rangle^{4\gamma+2s}F\|_1^2\Big(\|\langle\cdot\rangle^{(3\gamma+2s)/2}F_K^+\|_2^2 + K\,\|\langle\cdot\rangle^{\gamma}F_K^+\|_1 + K^2\|\langle\cdot\rangle^{3\gamma+2s}1_{\{F_K^+\geq 0\}}\|_1\Big)$$

where the constant c_0 *depends on* D_0, E_0, d, γ, *and* s. *The constant* C *depends also on the corresponding weight parameter* $k \geq 0$ *or* $r > 0$.

A similar argument to that of the L∞ - norm leads to

$$w(v) = \langle v\rangle^k \;\; or \; w(v) = e^{r\min\{t,1\}\langle v\rangle^{\gamma}}$$

$$\|w\,f(t)\|_\infty \leq C\left(\frac{1}{t^c}+1\right), \qquad t>0\,,$$

Emergence of exponentially weighted regularity: commutator

Theorem 5. *For* $\gamma \in [0,1]$, $s \in (0,1)$, *and* $w(v) = \langle v \rangle^k$ *or* $w(v) = e^{r\langle v \rangle^\gamma}$, *with* $k \geq 0$, $r \in (0, \frac{1}{2})$, *the following estimate holds*

$$\big(1 + (-\Delta)\big)^{\frac{s}{2}} Q(g, f) = Q(g, \big(1 + (-\Delta)\big)^{\frac{s}{2}} f) + \mathcal{R}(g, f)\,.$$

The remainder satisfies the estimate

$$\|w\,\mathcal{R}(g,f)\|_2 \leq C\|\sin^2\theta\, b\|_1 \Big(\|\langle\cdot\rangle^a\, w\, g\|_1 + \|w\, g\|_2\Big)\|\langle\cdot\rangle^a\, w\, f\|_{H^s}\,,$$

with $a = \max\{2\gamma, \gamma - 1 + 2s\}$. *The constant* C *depends, in addition to* d, s, *and* γ, *on the corresponding weight parameter* k *or* r *as well.*

This is an pointwise remainder result

Bobylev's formula

$$\mathcal{F}\Big\{(1+(-\Delta))^{\frac{s}{2}}Q(g,f)(v)\Big\}(\xi) = \langle\xi\rangle^s\mathcal{F}\Big\{Q(g,f)(v)\Big\}(\xi)$$
$$= \int_{\mathbb{S}^{d-1}}\Big(\langle\xi^+\rangle^s\mathcal{F}\{g(v_*)f(v)|u|^\gamma\}(\xi^-,\xi^+) - \langle\xi\rangle^s\mathcal{F}\{g(v_*)f(v)|u|^\gamma\}(0,\xi)\Big)b(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma$$
$$+ \int_{\mathbb{S}^{d-1}}\Big(\langle\xi\rangle^s - \langle\xi^+\rangle^s\Big)\mathcal{F}\{g(v_*)f(v)|u|^\gamma\}(\xi^-,\xi^+)b(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma =: \widehat{\mathcal{J}(g,f)}(\xi) + \widehat{\mathcal{I}(g,f)}(\xi)\,.$$

For the second term, it have been proven that

$$\big\|w\,\mathcal{I}(g,f)\big\|_2 \le C\|\sin^2\theta\, b\|_{L^1(\mathbb{S}^{d-1})}\Big(\|\langle\cdot\rangle^\gamma G\|_1 + \|G\|_2\Big)\|\langle\cdot\rangle^\gamma F\|_{H^s}\,.$$

Note that

$$-\Big((1+|\xi^+|^2)^{\frac{s}{2}} - \big(a(\hat{\xi}\cdot\sigma)+|\xi^+|^2\big)^{\frac{s}{2}}\Big) = -\frac{s}{2}\int_0^1 \frac{1-a(\hat{\xi}\cdot\sigma)}{\big((1-\theta)a(\hat{\xi}\cdot\sigma)+\theta+|\xi^+|^2\big)^{\frac{2-s}{2}}}\mathrm{d}\theta$$

Then

$$\widehat{\mathcal{I}_1^+(f,g)}(\xi) = -\frac{s}{2}\int_0^1\int_{\mathbb{S}^{d-1}}\Big\langle\frac{\xi^+}{\sqrt{\ell(\theta,\hat{\xi}\cdot\sigma)}}\Big\rangle^{-(2-s)}\widehat{F}(\xi^+,\xi^-)\frac{\big(1-a(\hat{\xi}\cdot\sigma)\big)b_s(\hat{\xi}\cdot\sigma)}{\ell(\theta,\hat{\xi}\cdot\sigma)^{\frac{2-s}{2}}}\mathrm{d}\sigma\mathrm{d}\theta\,.$$

$$\mathcal{I}_1^+(f,g)(v) = -\frac{s}{2}\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\varphi\big(\ell(\theta,\hat{u}\cdot\sigma)^{\frac{1}{2}}\,\cdot\,\big)*\big(f(\cdot)\tau_{v'_*}|\cdot|^\gamma\big)(v')\,g(v'_*)$$
$$\times\,\ell(\theta,\hat{u}\cdot\sigma)^{\frac{d+s-2}{2}}\big(1-a(\hat{u}\cdot\sigma)\big)b_s(\hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}\theta\,.$$

Complete cancelation

Fractional differentiation product

$$\langle\xi^+\rangle^s \mathcal{F}\big\{g(v_*)f(v)|u|^\gamma\big\}(\xi^-,\xi^+) = \mathcal{F}\Big\{g(v_*)\big(1+(-\Delta)\big)^{\frac{s}{2}}\big(f(\cdot)\tau_{v_*}|\cdot|^\gamma\big)(v)\Big\}(\xi^-,\xi^+)$$
$$= \mathcal{F}\Big\{g(v_*)\big(1+(-\Delta)\big)^{\frac{s}{2}} f(v)\,|u|^\gamma\big) + \mathcal{R}_{v_*}f(v)\Big\}(\xi^-,\xi^+)\,.$$

Leads to

$$\widehat{\mathcal{J}(g,f)}(\xi) = \int_{\mathbb{S}^{d-1}} \Big(\mathcal{F}\Big\{g(v_*)\big(1+(-\Delta)\big)^{\frac{s}{2}} f(v)\,|u|^\gamma\big)(v)\Big\}(\xi^-,\xi^+)$$
$$- \mathcal{F}\Big\{g(v_*)\big(1+(-\Delta)\big)^{\frac{s}{2}} f(v)\,|u|^\gamma\big)(v)\Big\}(0,\xi)\Big)b(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma$$
$$+ \int_{\mathbb{S}^{d-1}} \Big(\mathcal{F}\big\{\mathcal{R}_{v_*}f(v)\big\}(\xi^-,\xi^+) - \mathcal{F}\big\{R_{v_*}f(v)\big\}(0,\xi)\Big)b(\hat{\xi}\cdot\sigma)\mathrm{d}\sigma$$
$$=: \widehat{\mathcal{J}_1(g,f)}(\xi) + \widehat{\mathcal{J}_2(g,f)}(\xi)\,.$$

Explicit remainder

We already computed

$$\mathcal{R}_{v_*} f(v) = s \int_{\mathbb{R}^d} \tau_x f(v)\, \Phi(u,x)\, \frac{|\nabla \mathcal{B}(x)|}{|x|}\, \mathrm{d}x\,, \quad \text{where} \quad \Phi(u,x) = x \cdot \int_0^1 \tau_{\theta x}\Big(|u|^{\gamma-2} u\Big) \mathrm{d}\theta\,,$$

$$\mathcal{B} \;:=\; \mathcal{F}^{-1}\{\langle\cdot\rangle^{s-2}\}$$

Leading to

$$\mathcal{J}_2(g,f)(v) = s \int_{\mathbb{R}^d} \frac{|\nabla \mathcal{B}(x)|}{|x|}\, Q_{\Phi(u,x)}(g, \tau_x f)(v)\, \mathrm{d}x\,.$$

After some painful calculations and a classical result

Lemma 3. *For $\gamma \in (0,1]$ and $s \in (0,1)$ it follows that*

$$\int_{\mathbb{R}^d} \mathcal{J}_2(g,f)(v) h(v) dv \le C\Big(\|\langle\cdot\rangle^a g\|_1 + \|g\|_2\Big)\|\langle\cdot\rangle^a f\|_{H^s} \|h\|_{H^s}\,, \quad a = (\gamma - 1 + 2s)^+\,,$$

for some constant C depending only on d, s, and γ. In fact, for hard spheres $\gamma = 1$ the L^2-norm of g can be omitted.

Adding weights

$$\int_{\mathbb{R}^d} |\nabla \mathcal{B}(x)| \langle x \rangle^{k'} w(x) \mathrm{d}x < \infty \,, \qquad k' \in \mathbb{R} \,, \quad r \in \left(0, \tfrac{1}{2}\right) ,$$

Leads to

Corollary 2. *For $\gamma \in (0,1]$ and $s \in (0,1)$ it follows that*

$$\int_{\mathbb{R}^d} \mathcal{J}_2(g,f)(v)\, w(v)\, h(v)\, dv$$
$$\leq C\Big(\|\langle\cdot\rangle^a\, w\, g\|_1 + \|w\, g\|_2 \Big) \|\langle\cdot\rangle^a\, w\, f\|_{H^s} \|h\|_{H^s} \,, \quad a = \max\{2\gamma, \gamma - 1 + 2s\} \,,$$

for some constant C depending only on d, s, and γ. It also depends on the corresponding weight parameter $k \geq 0$ or $r \in \left(0, \frac{1}{2}\right)$. For hard spheres, the case $\gamma = 1$, the L^2-norm of g can be omitted.

Generation and propagation of smoothness

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\,f\|^2_{H^s} = \int_{\mathbb{R}^d} w\big(1+(-\Delta)\big)^{\frac{s}{2}} f\, w\big(1+(-\Delta)\big)^{\frac{s}{2}} Q(f,f)\mathrm{d}v + \int_{\mathbb{R}^d} w(\partial_t w)\Big|\big(1+(-\Delta)\big)^{\frac{s}{2}} f\Big|^2 \mathrm{d}v$$

$$= \int_{\mathbb{R}^d} w\big(1+(-\Delta)\big)^{\frac{s}{2}} f\, w\, Q\big(f, \big(1+(-\Delta)\big)^{\frac{s}{2}} f\big)\mathrm{d}v$$

$$+ \int_{\mathbb{R}^d} w\big(1+(-\Delta)\big)^{\frac{s}{2}} f\, w\, \mathcal{R}\big(f,f\big)\mathrm{d}v + \int_{\mathbb{R}^d} w\,(\partial_t w)\,\Big|\big(1+(-\Delta)\big)^{\frac{s}{2}} f\Big|^2 \mathrm{d}v$$

Leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F\|^2_2 + \tfrac{c_0}{8}\|\langle\cdot\rangle^{\frac{\gamma}{2}} F\|^2_{H^s} \le C\Big(t^{-6a'/\gamma} + 1\Big)\,.$$

Then,

Proposition 4. *Let* $\gamma \in (0,1]$ *and* $s \in (0,1)$, $r \in \big(0, \min\{r_0, \frac{1}{2}\}\big)$. *Assume that* $f_0 \in \mathcal{U}(D_0, E_0)$. *Then, it follows that*

$$\|e^{r\,\min\{t,1\}\langle v\rangle^{\gamma}}\,\big(1+(-\Delta)\big)^{\frac{s}{2}} f\|^2_2 \le C\left(\frac{1}{t^c} + 1\right), \quad t > 0\,.$$

The constant C *depends on* D_0, E_0, d, s, *and* γ, *whereas the constant* $c > 0$ *depends only on* d, s, γ *and* r.

Sobolev smoothness

Theorem 6 (Appearance and propagation of exponentially weighted higher regularity). *Let* $\gamma \in (0,1]$ *and* $s \in (0,1)$, $k \in \mathbb{N}$, *and* $r \in \left(0, \min\{r_0, \frac{1}{2}\}\right)$. *Assume that* $f_0 \in \mathcal{U}(D_0, E_0)$. *Then, it follows that*

$$\|e^{r\,\min\{t,1\}\langle v\rangle^\gamma}\,\partial^k f\|_2^2 \le C_k\left(\frac{1}{t^{c_k}} + 1\right), \quad t > 0\,.$$

The constant C_k *depends on* D_0, E_0, d, s, *and* γ, *whereas the constant* $c_k > 0$ *depends only on* k, d, s, *and* γ. *Furthermore, in the range* $\gamma \in [0,1]$, *if* $e^{r_0\langle v\rangle^\alpha} f_0 \in L^1$ *and* $e^{r\langle v\rangle^\alpha}\partial^k f_0 \in L^2$, *with* $\alpha \in (0,1]$, *then*

$$\sup_{t\ge 0}\|e^{r\,\min\{t,1\}\langle v\rangle^\alpha}\,\partial^k f(t)\|_2 \le \max\left\{\|e^{r\langle v\rangle^\alpha}\partial^k f_0\|_2, C_k'\right\}.$$

The constant C_k' *depends additionally on the* L^1*-exponential norm of* f_0.

One key step: using Prop. 4

$$\int_{\mathbb{R}^d} Q(\partial^k f, f)\, w\, w\, \partial^k f \,\mathrm{d}v \leq C_{d,s,\gamma}\Big(\|\langle\cdot\rangle^a\, w\, \partial^k f\|_1 + \|w\,\partial^k f\|_2\Big)\|\langle\cdot\rangle^a\, w\, f\|_{H^s}\|w\,\partial^k f\|_{H^s}$$

$$\leq C_{d,s,\gamma}(t^{-c}+1)\|\langle\cdot\rangle^{a+d}\, w\,\partial^k f\|_2\|w\,\partial^k f\|_{H^s}\,.$$

Interpolation weight - regularity

$$\|\langle\cdot\rangle^{a+d}\, w\,\partial^k f\|_2 \leq C_s\|\langle\cdot\rangle^{n_s(a+d)}\, w\,\partial^{k-1} f\|_2^{1/n_s}\|w\,\partial^k f\|_{H^s}^{\theta_s}\,.$$

$$n_s > 0 \text{ and } \theta_s \in (0,1)$$

Leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\,\partial^k f\|_2^2 + c_0\|w\,\partial^k f\|_{H^s}^2 \leq C_{d,s,\gamma}\Big(t^{-\frac{2c'_k}{1-\theta_s}}+1\Big)\,,$$

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