On the maximum order complexity of subsequences of the Thue-Morse and Rudin-Shapiro sequence along squares

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UDT, Luminy, Oct. 2, 2018

How do we measure the suitability of a binary sequence for cryptography?

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How do we detect 'bad' cryptographic sequences?

How do we find 'good' cryptographic sequences?

How do we measure the suitability of a binary sequence for cryptography?

Study several quality measures:

- **linear complexity**
- **maximum order complexity**
- **correlation measure**
- Gowers norm
- expansion complexity
- ...

Studied sequences

- Thue-Morse sequence (Rudin-Shapiro sequence)
- pattern sequences
- (non-periodic) automatic sequences
- subsequence of Thue-Morse along the squares
- Legendre sequence (with polynomials)
- ...

Linear Complexity

For a positive integer N the Nth linear complexity $L(s_n, N)$ of a sequence (s_n) over $\mathbb{F}_2$ is the smallest positive integer L such that there are constants $c_0, \ldots, c_{L-1} \in \mathbb{F}$ satisfying

$$s_{n+L} = c_{L-1}s_{n+L-1} + \ldots + c_0 s_n,$$

$0 \leq n \leq N - L - 1.$

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$0 \leq n \leq N - L - 1$.

- small linear complexity implies predictability and thus unsuitability in cryptography
- there are predictable sequences with large linear complexity
- we need further measures for the unpredictability of a sequence

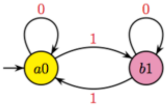
Automatic sequences:

- high linear complexity
- still predictable

2-automatic sequences

Thue-Morse sequence $s_0 = 0,\ s_{2n+1} = s_n + 1,\ s_{2n} = s_n,\ n = 0, 1, \dots$

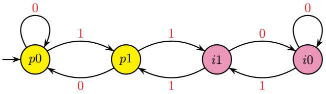
Input $n = (n_0 n_1 \dots n_j)_2$



Output:0, 1

$011010011001011 0\dots$ $n = 5 = (101)_2$

Rudin-Shapiro



$0001001000011101\dots$

Christol's theorem

generating function $G(x)$ of (s_n):

$$G(x) = \sum_{n=0}^{\infty} s_n x^n$$

(s_n) is 2-automatic over $\mathbb{F}_2$ if and only if $G(x)$ is algebraic over $\mathbb{F}_2[x]$.

Example. Thue-Morse sequence

$$(x+1)^3 G(x)^2 + (x+1)^2 G(x) + x = 0$$

$G(x)$ rational if and only if (s_n) is ultimately periodic.

Linear complexity of Thue-Morse sequence

Mérai/W., 2018:

$$t_{n+L} = \sum_{\ell=0}^{L-1} c_\ell t_{n+\ell} = 0, \quad 0 \le n \le N-1-L$$

$$f(x) = \sum_{n=0}^{L} c_\ell x^{L-n}, \quad c_L = 1$$

$$g(x) := f(x)G(x) \bmod x^N$$

$\deg(g) < L$, $\deg(f) \le L$

$$(x+1)^3 g(x)^2 + (x+1)^2 f(x) g(x) + x f(x)^2 = k(x) x^N, \quad k(x) \neq 0$$

$$L(t_n, N) \ge \frac{N-1}{2}$$

- A different method gives $L(t_n, N) = 2\lfloor (N+2)/4 \rfloor$.
- The method works for any (non-periodic) automatic sequence (if we know $h \neq 0$ with $h(x, G(x)) = 0$).
- What measure(s) can be used to see that the Thue-Morse sequence (other automatic sequences) is (are) predictable?
- What can we say about (non-automatic) subsequences of automatic sequences?
- Are there any sequences with good results under all commonly used measures of pseudorandomness?

Correlation measure of order k

Mauduit/Sárközy, 1997:
The correlation measure of order k of a binary sequence $(s_n)_{n=0}^{T-1}$ of length T is introduced as

$$C_k(s_n) = \max_{M,D} \left| \sum_{n=1}^{M} (-1)^{s_{n+d_1}} \cdots (-1)^{s_{n+d_k}} \right|, \quad k \geq 1,$$

where the maximum is taken over all $D = (d_1, d_2, \ldots, d_k)$ with non-negative integers $d_1 < d_2 < \cdots < d_k$ and M such that $M - 1 + d_k \leq T - 1$.

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Mauduit/Sárközy, 1998: $C_2(t_n) \geq N/12$, $N \geq 5$
Mérai/W. 2018: lower bounds on $C_2(s_n)$ for any automatic sequence.

Relation between linear complexity and correlation measure

Brandstätter/W., 2006:

$$L(s_n, N) \geq N - \max_{1 \leq k \leq L(s_n,N)+1} C_k(s_n), \quad 2 \leq N \leq T - 1.$$

Example: Legendre sequence

Let $p > 2$ be a prime. The *Legendre-sequence* (ℓ_n) is defined by

$$\ell_n = \begin{cases} 1, & \left(\frac{n}{p}\right) = -1, \\ 0, & \text{otherwise}, \end{cases} \qquad n \geq 0,$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre-symbol.

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The Nth linear complexity of the Legendre sequence satisfies

$$L(\ell_n, N) \gg \frac{\min\{N, p\}}{p^{1/2} \log p}, \quad N \geq 1.$$

$$L(s_n,N)\geq N-\max_{1\leq k\leq L(s_n,N)+1} C_k(s_n),\quad 2\leq N\leq T-1.$$

Mauduit/Sárközy, 1997:

$$C_k(\ell_n)\ll kp^{1/2}\log p$$

$$L(\ell_n,N)\gg \frac{\min\{N,p\}}{p^{1/2}\log p},\quad N\geq 1.$$

Maximum order complexity

Maximum order complexity: Smallest $M = M(s_n, N)$ with

$$s_{n+M} = f(s_{n+M-1}, \ldots, s_n), \quad 0 \leq n \leq N - M - 1.$$

- finer than linear complexity
- much more complicated to calculate

Maximum order complexity vs. correlation measure

Isik, W., 2017: For any binary sequence (s_n) we have

$$M(s_n,N) \geq N - 2^{M(s_n,N)+1} \max_{1\leq k\leq M(s_n,N)+1} C_k(s_n,N), \quad N \geq 1.$$

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Example. Legendre sequence

$$C_k(\ell_n) \ll kp^{1/2} \log p$$

$$M(\ell_n, N) \geq \log(\min\{N, p\}/p^{1/2}) + O(\log \log p)$$

typical: $M(s_n, N) \approx \log N$, $C_k(s_n) \approx N^{1/2} \log^{c(k)} N$

Maximum-order complexity of Thue-Morse sequence

Sun/W., submitted: $M(t_n) \geq N/5$

- Thue-Morse sequence is predictable since $C_2(t_n)$ is large.
- still $M(t_n)$ is very large

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- still $M(t_n)$ is very large

- Do certain subsequences of the Thue-Morse sequence inherit a large maximum order complexity?
- For example, the Thue-Morse sequence along the squares is not automatic and normal (Drmota/Mauduit/Rivat).

Thue-Morse sequence along squares

$s_n = t_{n^2},\ n = 0, 1, \ldots$

Sun/W., submitted: $M(s_n) \gg N^{1/2}$

Idea of proof: $5 \cdot 2^\ell < N \leq 5 \cdot 2^{\ell+1}$
Verify

$$t_{(i+2^{\ell+1})^2} = t_{(i+2^{\ell+2})^2}, \quad i = 0, 1, \ldots, \left\lfloor \sqrt{2^{\ell+2} - 1} \right\rfloor$$

and

$$t_{(2^\ell+2^{\ell+1})^2} \neq t_{(2^\ell+2^{\ell+2})^2}.$$

Assume $M \leq \left\lfloor \sqrt{2^{\ell+2} - 1} \right\rfloor + 1$ such that

$$t_{(i+M)^2} = f(t_{i^2}, \ldots, t_{(i+M-1)^2}), \quad i = 0, 1, \ldots, N - M - 1$$

we get a contradiction.

A generalization

The pattern sequences $\mathcal{P}_k = (p_n)_{n=0}^{\infty}$ is defined by

$$p_n \equiv s_k(n) \bmod 2,$$

where $P_k = 11\ldots1 \in \mathbb{F}_2^k$ is the all 1 pattern of length k and $s_k(n)$ is the number of occurrences of P_k in the binary representation of n.

$k = 1$: Thue-Morse sequence
$k = 2$: Rudin-Shapiro sequence

$$M(\mathcal{P}_k', N) \gg N^{1/2}, \quad N \geq 2^{2k+2}.$$

Thank you for your attention!