Growth and geometry in $SL(2,\mathbb{Z})$ dynamics

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- Hyperbolic geometry: discrete version of $PSL_2(\mathbb{R})$

Corresponding tessellation of $\mathbb{H}^2$ is known as **Farey tesselation**.
Farey "addition" (mediant): $\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d}$.

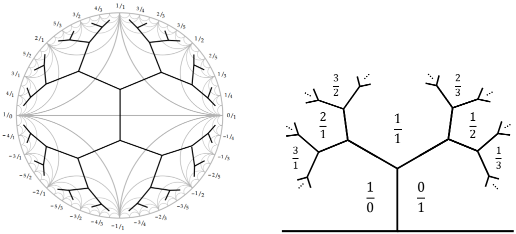

Figure: Dual tree for Farey tessellation and Farey tree

Markov spectrum of real numbers

Markov constant of a real number α is the minimal possible c such that

$$|\alpha - \frac{p}{q}| < \frac{c}{q^2}$$

holds for infinitely many p, q:

$$\mu(\alpha) = \liminf_{N\to\infty}([a_{N+1}, a_{N+2}, \ldots] + [0; a_N, a_{N-1}, \ldots, a_1])^{-1}.$$

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α	$\mu(\alpha)$	
$\frac{1+\sqrt{5}}{2}$	$\frac{1}{\sqrt{5}}$	$=0.4472135\ldots$
$1+\sqrt{2}$	$\frac{1}{\sqrt{8}}$	$=0.3535533\ldots$
$\frac{9+\sqrt{221}}{10}$	$\frac{5}{\sqrt{221}}$	$=0.3363363\ldots$
$\frac{23+\sqrt{1517}}{26}$	$\frac{13}{\sqrt{1517}}$	$=0.3337725\ldots$
$\frac{5+\sqrt{7565}}{58}$	$\frac{29}{\sqrt{7565}}$	$=0.3334214\ldots$

Table: The top five most irrational numbers and their Markov constants

Markov triples and Markov Theorem

Markov numbers are parts of the **Markov triples,** which are the positive integer solutions of the **Markov equation**

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$$\sigma : (x, y, z) \to (x, y, 3xy - z)$$

and permutations:

$$(1, 1, 1) \to (1, 1, 2) \to (1, 2, 1) \to (1, 2, 5) \to (1, 5, 2) \to (1, 5, 13) \to \ldots$$

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Markov (1880): Markov spectrum above $1/3$ is discrete and consists of

$$\mu = \frac{m}{\sqrt{9m^2 - 4}},$$

where m is one of the Markov numbers:

$$m = 1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985, \dots$$

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The most irrational number corresponding to triple (x, y, z) is

$$\alpha = \frac{b}{x} + \frac{y}{xz} - \frac{3}{2} + \frac{\sqrt{9z^2 - 4}}{2z}, \quad by - ax = z.$$

Growth of the Markov numbers

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However, we have a natural action of $PSL_2(\mathbb{Z}) \approx \mathbb{Z}_2 * \mathbb{Z}_3$ generated by Markov involution and cyclic permutation, so Markov triples can be shown on the tree:

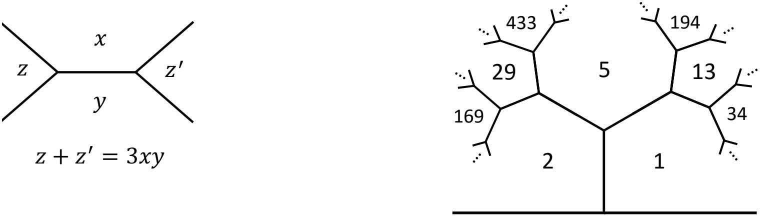

Figure: Vieta involution and Markov tree

Question. What is the growth along a path on the Markov tree?

Markov, Euclid and Farey trees

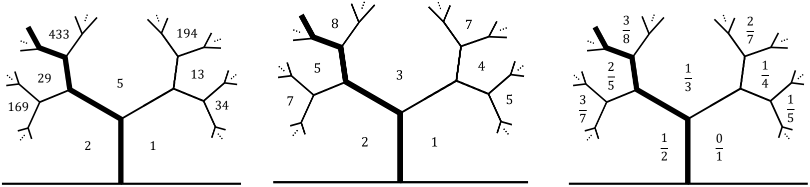

Figure: Markov, Euclid and Farey trees with the "golden" path

Paths $\gamma = \gamma(x)$ on Farey (and thus on Markov) tree can be labelled by $x \in \mathbb{R}P^1$ using continued fraction expansion

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \ddots}} := [c_0, c_1, c_2, \ldots] \,.$$

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Define the **Lyapunov exponent** $\Lambda(x)$ as

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where $m_n(x)$ is n-th Markov number along the path $\gamma(x)$.

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It can be shown that

$$\Lambda(x) = \limsup_{n \to \infty} \frac{\ln c_n(x)}{n},$$

where $c_n(x)$ is the corresponding number on the Euclid tree.

Lyapunov spectrum of Markov tree

K. Spalding, AV (2017):

- *$\Lambda(x)$ is defined for all $x \in \mathbb{R}P^1$ and is $GL_2(\mathbb{Z})$-invariant:*

$$\Lambda\left(\frac{ax+b}{cx+d}\right) = \Lambda(x), \quad x \in \mathbb{R}P^1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$$

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$$Spec_\Lambda = [0, \ln \varphi],$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ *is the golden ratio.*

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- *The restriction of* Λ *on the Markov-Hurwitz set* X *of the most irrational numbers is monotonically increasing from* $\Lambda(\sqrt{2}) = \frac{1}{2}\ln(1+\sqrt{2})$ *to* $\Lambda(\varphi) = \ln\varphi$ *and in the Farey parametrization is convex.*

Proof is based on results from hyperbolic geometry by Fricke and Klein, Gorshkov, Cohn, V. Fock.

Crucial link: geodesics on punctured torus

Klein, Poincare: **Uniformization Theorem**
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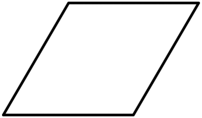
Let T^2_* be the punctured "equianharmonic" torus with hyperbolic metric



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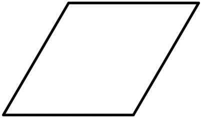

Crucial observation (Gorshkov, 1953, Cohn, 1955):

Markov numbers m are related to the lengths l of simple geodesics on T^2_* by the formula

$$m = \frac{2}{3} \cosh \frac{l}{2}.$$

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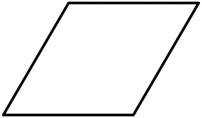

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Action of mapping class group $SL_2(\mathbb{Z})$ is generated by cyclic permutations and Markov involutions.

Let A, B be the generators of the corresponding Fuchsian subgroup Γ of $SL_2(\mathbb{R})$.

Fricke: for any $A, B \in SL_2(\mathbb{R})$ and $C = AB$ we have

$$(tr\,A)^2 + (tr\,B)^2 + (tr\,C)^2 = tr\,A\,tr\,B\,tr\,C + tr\,(ABA^{-1}B^{-1}) + 2.$$

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$X = tr\, A$, $Y = tr\, B$, $Z = tr\, C$ satisfy the real Markov equation

$$X^2 + Y^2 + Z^2 = XYZ, \quad X, Y, Z \in \mathbb{R},$$

which defines the *Teichmüller space of one-punctured tori* (Fricke and Klein, Keen et al).

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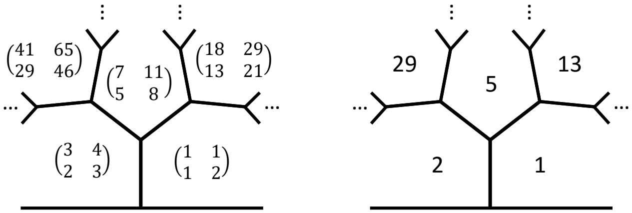
Markov orbit corresponds to the punctured equianharmonic torus T^2_*.

H. Cohn (1955): replace $a + b = c$ by $AB = C$:



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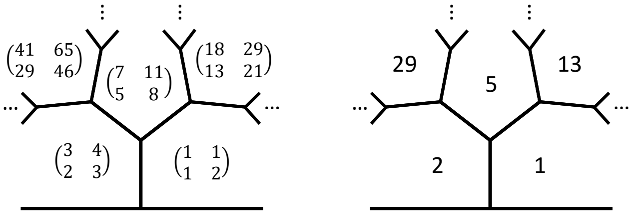

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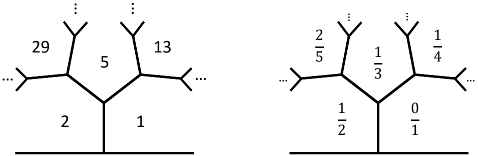
Key fact: Cohn matrices A and B generate the Fuchsian group $\Gamma = SL_2(\mathbb{Z})'$ giving explicit uniformization of T^2_* as the quotient $\mathbb{H}^2/\Gamma$.

Fock's theorem

Let $m(\frac{p}{q})$ be the Markov number corresponding to $\frac{p}{q}$ on Farey tree and define the function

$$\psi(\frac{p}{q}) = \frac{1}{q}\cosh^{-1}\left(\frac{3}{2}m(\frac{p}{q})\right).$$



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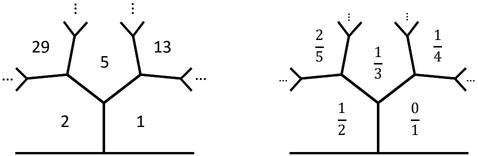

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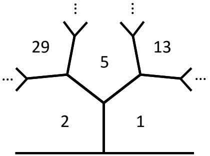

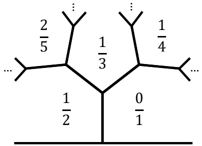

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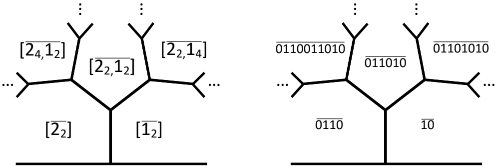
Sorrentino, AV (2017): relation with the theory of Federer-Gromov's *stable norm*.

Markov-Hurwitz and Minkowski trees

Let $x\left(\frac{p}{q}\right)$ be the "most irrational" number corresponding to $m = m\left(\frac{p}{q}\right)$.
Key observation:

$$\Lambda(x(\frac{p}{q})) = \frac{1}{2}\psi(\frac{p}{q}).$$



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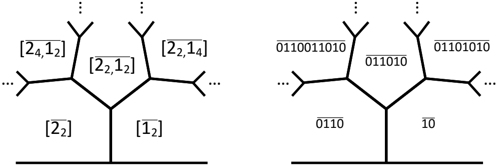

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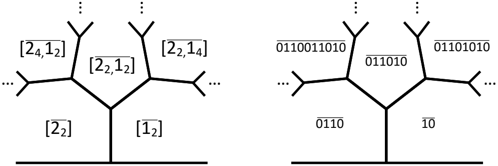

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Spalding, AV (2017): generalisation to modified Markov equation

$$x^2 + y^2 + z^2 = xyz + 4 - 4a^6, \; a \in \mathbb{N}.$$

Conway's topograph

Conway (1997): "topographic" way to "vizualise" the values of a binary quadratic form

$$Q(x, y) = ax^2 + hxy + by^2, \quad (x, y) \in \mathbb{Z}^2$$

by taking values of Q on the lax vectors of superbases $e_1 + e_2 + e_3 = 0$:

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One can construct the *topograph* of Q using *Arithmetic progression rule*:

$$Q(\mathbf{u} + \mathbf{v}) + Q(\mathbf{u} - \mathbf{v}) = 2(Q(\mathbf{u}) + Q(\mathbf{v})), \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^2.$$

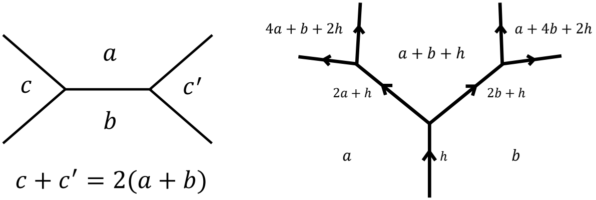

Figure: Arithmetic progression rule and Conway's Climbing Lemma.

Euclidean example

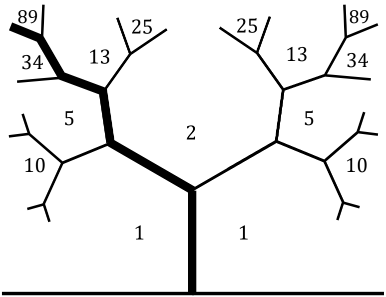

Figure: Topograph of $Q = x^2 + y^2$ and Farey tree with marked "golden" path.

For indefinite binary quadratic form $Q(x,y)$ the situation is more interesting: positive and negative values of Q are separated by the path on the topograph called **Conway river**. For integer form Q the Conway river is periodic.

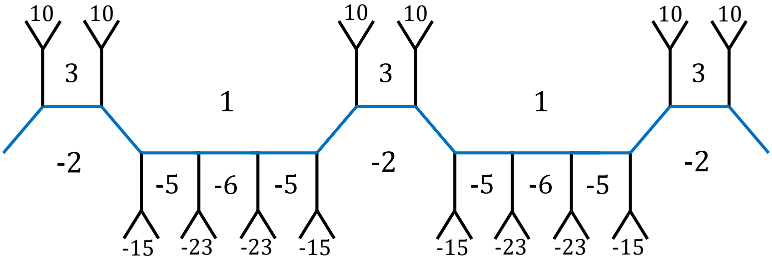

Figure: Conway river for the quadratic form $Q = x^2 - 2xy - 5y^2$.

Define

$$\Lambda_Q(\xi) = \limsup_{n\to\infty} \frac{\ln |Q_n(\xi)|}{n}, \quad |Q_n(\xi)| = \max(|a_n(\xi)|, |b_n(\xi)|, |c_n(\xi)|).$$

Growth of values of quadratic forms

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Spalding, AV (2017): *For an indefinite form Q not representing zero*

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \quad \xi \neq \alpha_\pm$$

with $\Lambda_Q(\alpha_\pm) = 0 \neq 2\Lambda(\alpha_\pm)$.

Define

$$\Lambda_Q(\xi) = \limsup_{n\to\infty} \frac{\ln|Q_n(\xi)|}{n}, \quad |Q_n(\xi)| = \max(|a_n(\xi)|, |b_n(\xi)|, |c_n(\xi)|).$$

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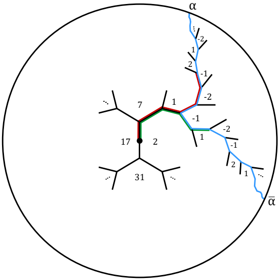
In other words, the only exceptional paths are those following Conway river:



Some open questions and references

- Further study of $\Lambda(x)$, in particular generalisations of Markov-Hurwitz sets (cf. Karpenkov, Van-Son, 2018)

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