An elementary approach to Somos-4 sequences

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Somos–(4)

Somos–(4) sequence $\{s_n\}$ is defined by initial data

$$s_1 = s_2 = s_3 = s_4 = 1$$

and recurrence relation

$$s_{n+2}s_{n-2} = s_{n+1}s_{n-1} + s_n^2.$$

It begings with

$$\ldots, 2, 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, \ldots$$

(Obviously $s_n = s_{5-n}$.)

Somos–(6)

Somos first introduced the sequence Somos-(6) such that

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 1$$

and

$$s_{n+3}s_{n-3} = s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

He raised the question whether all the terms are integer:

1470*. *Proposed by Michael Somos, Cleveland, Ohio.*

Consider the sequence (a_n) where $a_0 = a_1 = \cdots = a_5 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2}{a_{n-6}}$$

for $n \geq 6$. Computer calculations show that $a_6, a_7, \ldots, a_{100}$ are all integers. Consequently it is conjectured that all the a_n are integers. Prove or disprove.

Somos M. Problem 1470. Crux Mathematicorum, 15: 7 (1989), p. 208.

Integrality

The integrality of Somos–(4) and Somos–(5) was proved by Janice Malouf, Enrico Bombieri and Dean Hickerson (1990).

The integrality of Somos–(6) was proved by Dean Hickerson (April 1990).

The integrality of Somos–(7) was proved by Ben Lotto (May 1990).

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Somos–(8)

Somos–(8) is a sequence with initial data

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7 = s_8 = 1$$

satisfying recurrence relation

$$s_{n+4}s_{n-4} = s_{n+3}s_{n-3} + s_{n+2}s_{n-2} + s_{n+1}s_{n-1} + s_n^2.$$

Somos–(8) is NOT an integer sequence:

$$\dots, 1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61, 187, 775, 5827, 14815, \frac{420514}{7}, \dots$$

[It is a wild object with no properties.]

Somos sequences

Definition

For integer $k \geq 4$ Somos–k sequence is a sequence generated by quadratic recurrence relation of the form

$$s_{n+k}s_n = \sum_{j=1}^{[k/2]} \alpha_j s_{n+k-j}s_{n+j},$$

where α_j are constants and $s_0, \ldots, s_{k-1}$ are initial data.

In particular Somos-4 is defined by initial data s_0, s_1, s_2, s_3 and fourth-order recurrence

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2,$$

Somos-6 is defined by initial data $s_0, \ldots, s_5$ and sixth-order recurrence

$$s_{n+3}s_{n-3} = \alpha s_{n+2}s_{n-2} + \beta s_{n+1}s_{n-1} + \gamma s_n^2.$$

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A sigma-function solution for Somos–4

The integrality of Somos–(4)$\cdots$(7) may be proved by elementary methods. But elementary proofs don't spread any light on the nature of Somos sequences: there is some elliptic curve hidden behind Somos–4 and hyperelliptic curve of genus 2 behind Somos–6.

A solution of general Somos–4 recurrence relation was given by C. Swart (2003) and A. Hone (2005)

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where z, $z_0 \in \mathbb{C}^*$, and

$$\sigma(z) = \sigma_\Gamma(z) = z \prod_{w \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2}$$

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Integrality condition

Theorem (Fomin and Zelevinsky, 2002)

For a Somos–k sequences ($k = 4, 5, 6$ and 7) all of the terms in the sequences are Laurent polynomials in these initial data whose coefficients are in $\mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}]$, so that

$$s_n \in \mathbb{Z}[\alpha_1, \dots, \alpha_{[k/2]}, s_1^{\pm 1}, \dots, s_k^{\pm 1}] \text{ for all } n \in \mathbb{Z}.$$

Fomin S. and Zelevinsky A. “The Laurent Phenomenon”, Adv. Appl. Math. 28 (2002) 119–144.

Integrality of original Somos–(k) sequences follows from the theorem with

$$\alpha_1 = \dots = \alpha_{[k/2]} = s_1 = \dots = s_k = 1.$$

But this Theorem is not a final step.

Finite rank sequences

The main property of Somos–4 and Somos–6: they have finite rank.

Definition

The sequence $\{s_n\}_{n=-\infty}^{\infty}$ has a (finite) rank r if maximal rank of two infinite matices

$$\left(s_{m+n}s_{m-n}\right)\Big|_{m,n=-\infty}^{\infty}, \qquad \left(s_{m+n+1}s_{m-n}\right)\Big|_{m,n=-\infty}^{\infty}$$

is r.

Definition (1)

The sequence $\{s_n\}_{n=-\infty}^{\infty}$ has a (finite) rank r if r is a least possible k such that for all integer m and n

$$s_{m+n}s_{m-n} = \sum_{j=1}^{k} f_j(m)g_j(n), \qquad s_{m+n+1}s_{m-n} = \sum_{j=1}^{k} \widetilde{f}_j(m)\widetilde{g}_j(n).$$

Finite rank sequences: examples

Example (1)

Let $s_n = AB^n$. Then

$$s_{m+n}s_{m-n} = A^2B^{2m} = f(m)g(n),$$

where

$$f(m) = A^2B^{2m} \quad \text{and} \quad g(n) = 1.$$

(And almost the same for $s_{m+n+1}s_{m-n}$.) So the sequence $s_n = AB^n$ has rank 1.

Example (2)

The sequence $s_n = n$ has rank 2 because

$$s_{m+n}s_{m-n} = m^2 - n^2, \quad \text{and} \quad s_{m+n+1}s_{m-n} = m(m+1) - n(n+1).$$

Finite rank sequences: examples

Example (3)

$s_n = n^k$ has rank $2k$.

Example (4)

$s_n = \sin n$ has rank 2: $\sin(m+n)\sin(m-n) = \frac{1}{2}(\cos 2m - \cos 2n)$.

Example (5)

For Fibonacci sequence $s_n = F_n$ we have

$$F_{m+n}F_{m-n} = \frac{1}{5}\left(L_{2m} - (-1)^{m+n}L_{2n}\right),$$
$$F_{m+n+1}F_{m-n} = \frac{1}{5}\left(L_{2m+1} - (-1)^{m+n}L_{2n+1}\right),$$

where $L_n = F_{n-1} + F_{n+1}$ are Lucas numbers. So Fibonacci sequence has rank 2 as well.

Finite rank sequences: non-trivial examples

Example (6)

The sequence $s_n = \sigma(z_0 + zn)$ has rank 2 because of addition formula

$$\sigma(u - v)\sigma(u + v) = -\wp(u)\sigma(u)^2\sigma(v)^2 + \wp(v)\sigma(u)^2\sigma(v)^2.$$

Example (7)

The same for

$$s_n = \theta_j(z_0 + zn)$$

because we know addition theorems of the form

$$\theta_1(y + z)\theta_1(y - z)\theta_4^2 = \theta_3^2(y)\theta_2^2(z) - \theta_2^2(y)\theta_3^2(z) \ldots$$

Finite rank sequences: Somos examples

Example (Ma, 2010; conjectured by Gosper & Schroeppel, 2007)

Somos-4 has rank 2.

It follows from general formula

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}}.$$

Example (Hone, 2016; conj. by Gosper & Schroeppel, 2007)

Somos-6 has rank 4.

It follows from general formula for s_n in terms of the Kleinian sigma-function of genus two.

Somos-4: the main property
Magic determinant

General Somos-4 has rank 2. It means that Somos–4 sequence satisfy "magic determinant property" or "addition formula" (Ma, 2010)

$$\begin{array}{cc} & \begin{array}{ccc} \qquad n_1 \qquad & \qquad n_2 \qquad & \qquad n_3 \qquad \end{array} \\ \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} & \begin{vmatrix} s_{m_1+n_1}s_{m_1-n_1} & s_{m_1+n_2}s_{m_1-n_2} & s_{m_1+n_3}s_{m_1-n_3} \\ s_{m_2+n_1}s_{m_2-n_1} & s_{m_2+n_2}s_{m_2-n_2} & s_{m_2+n_3}s_{m_2-n_3} \\ s_{m_3+n_1}s_{m_3-n_1} & s_{m_3+n_2}s_{m_3-n_2} & s_{m_3+n_3}s_{m_3-n_3} \end{vmatrix} = 0, \end{array}$$

where m_i, n_i ($i = 1, 2, 3$) are arbitrary integers or half-integers. It is the main property because another properties of Somos-4 sequence follow from "magic determinant".

The identity

$$\begin{array}{cc} & \begin{array}{ccc} k & 1 & 0 \end{array} \\ \begin{array}{c} n \\ 1 \\ 0 \end{array} & \begin{vmatrix} s_{n+k}s_{n-k} & s_{n+1}s_{n-1} & s_n^2 \\ s_{1+k}s_{1-k} & s_2s_0 & s_1^2 \\ s_ks_{-k} & s_1s_{-1} & s_0^2 \end{vmatrix} \end{array} = 0,$$

means that Somos-4 is Somos-k for arbitrary $k \geq 4$: for some α_k, β_k

$$s_{n+k}s_{n-k} = \alpha_k s_{n+1}s_{n-1} - \beta_k s_n^2.$$

General "magic determinant property" follows from this equation by simple algebraic manipulations.

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$$\begin{array}{cc} & \begin{array}{ccc} \color{red}{k} & \color{red}{1} & \color{red}{0} \end{array} \\ \begin{array}{c} \color{red}{n} \\ \color{red}{1} \\ \color{red}{0} \end{array} & \begin{vmatrix} s_{n+k}s_{n-k} & s_{n+1}s_{n-1} & s_n^2 \\ s_{1+k}s_{1-k} & s_2s_0 & s_1^2 \\ s_ks_{-k} & s_1s_{-1} & s_0^2 \end{vmatrix} = 0, \end{array}$$

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The main goal

For Somos-4 sequence we must have

$$s_{n+k}s_{n-k} = \alpha_k s_{n+1}s_{n-1} - \beta_k s_n^2,$$
$$s_{n+k+1}s_{n-k} = \widetilde{\alpha}_k s_{n+2}s_{n-1} - \widetilde{\beta}_k s_{n+1}s_n,$$

Purely algebraic proof of this formula was given by van der Poorten and Swart (2006). But it was based on some tricky symmetry.

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Gauge transformations: 2 degrees of freedom

Somos-4 sequences

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2$$

are invariant under the two-parameter abelian group of gauge transformations defined by

$$s_n \to \widetilde{s}_n = A \cdot B^n \cdot s_n.$$

Thus it is natural to introduce the gauge-invariant variables

$$f_n = \frac{s_{n-1}s_{n+1}}{s_n^2}$$

satisfying

$$f_{n+1}f_n^2 f_{n-1} = \alpha f_n + \beta.$$

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First non-trivial case

The equation

$$\begin{vmatrix} s_{n+3}s_{n-2} & s_{n+2}s_{n-1} & s_{n+1}s_n \\ s_4 s_{-1} & s_3 s_0 & s_2 s_1 \\ s_3 s_{-2} & s_2 s_{-1} & s_1 s_0 \end{vmatrix} = 0$$

is equivalent to

$$f_n f_{n-1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n-1}}\right) + \frac{\beta}{f_n f_{n-1}} = f_2 f_1 + \alpha\left(\frac{1}{f_2} + \frac{1}{f_1}\right) + \frac{\beta}{f_2 f_1}.$$

So the quantity

$$T = f_n f_{n-1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n-1}}\right) + \frac{\beta}{f_n f_{n-1}}$$

must be conserved. (This equation defines an elliptic curve in the (f_n, f_{n-1}) plane.)

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Invariance of the first integral T

$$f_{n+1} f_n^2 f_{n-1} - \alpha f_n = \beta = f_{n+2} f_{n+1}^2 f_n - \alpha f_{n+1} \qquad /(f_n f_{n+1})$$

$$f_{n-1} f_n - \frac{\alpha}{f_{n+1}} = \frac{\beta}{f_n f_{n+1}} = f_{n+1} f_{n+2} - \frac{\alpha}{f_n} \qquad + f_n f_{n+1} + \alpha \left(\frac{1}{f_n} + \frac{1}{f_{n+1}} \right)$$

$$f_{n-1} f_n + f_n f_{n+1} + \frac{\alpha}{f_n} = T = f_n f_{n+1} + f_{n+1} f_{n+2} + \frac{\alpha}{f_{n+1}}$$

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$$f_{n-1}f_n+f_nf_{n+1}+\frac{\alpha}{f_n}=T=f_nf_{n+1}+f_{n+1}f_{n+2}+\frac{\alpha}{f_{n+1}}$$

In particular this proof gives new representation for T

$$f_n^2(f_{n-1}+f_{n+1})=Tf_n-\alpha$$

Together with the main equation

$$f_{n+1}f_n^2f_{n-1}=\alpha f_n+\beta$$

it allows to express $f_{n-1}+f_{n+1}$ and $f_{n-1}f_{n+1}$ in terms of f_n.

Induction

Desired formula

$$s_{n+k}s_{n-k} = \alpha_k s_{n+1}s_{n-1} - \beta_k s_n^2$$

can be rewritten as

$$\frac{s_{n+k}s_{n-k}}{s_n^2} = \alpha_k f_n - \beta_k.$$

Induction step:

$$s_{n+k}s_{n-k} = \frac{(s_{n+k}s_{n-k+2})(s_{n-k}s_{n+k-2})}{s_{n+k-2}s_{n-k+2}}.$$

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RHS as a symmetric function of f_{n+1} and f_{n-1} can be expressed as a function of f_n only

$$(\alpha_{k+1}f_n - \beta_{k+1})(\alpha_{k-1}f_n - \beta_{k-1}) = \beta_k^2 f_n^2 + \alpha_k(\alpha_k\beta - \beta_k T)f_n + \alpha_k(\alpha_k\beta + \alpha\beta_k)$$

Comparing coefficients from both sides of this formula we get recurrences for α_k and β_k.

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Comparing coefficients from both sides of this formula we get recurrences for α_k and β_k.

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The coefficients of f_n^0 and f_n^2 give recurrences

$$\alpha_{k+1}\alpha_{k-1} = \beta_k^2, \qquad \beta_{k+1}\beta_{k-1} = \alpha_k(\beta\alpha_k + \alpha\beta_k).$$

The coefficient of f_n^1 give extra condition on α_k and β_k. In terms of variables $g_k = \beta_k/\alpha_k$ it can be written as

$$g_n g_{n-1} + \alpha\left(\frac{1}{g_n} + \frac{1}{g_{n-1}}\right) + \frac{\beta}{g_n g_{n-1}} = T$$

But we know that this condition follows from recurrence

$$g_{k+1}g_{k-1} = \frac{1}{g_k^2}\left(\alpha g_k + \beta\right)$$

which is indeed the case.

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Elliptic divisibility sequences

We can associate with $g_k = \beta_k/\alpha_k$ a Somos-4 sequence $\{W_k\}$ such that $g_k = \frac{W_{k-1}W_{k+1}}{W_k^2}$. It is sufficient to take $W_{k-1}W_{k+1} = \beta_k$ and $W_k^2 = \alpha_k$ (small ambiguousness can be resolved via initial conditions).
This sequence is known as Elliptic divisibility sequence.
Additional formula for general Somos-4

$$s_{n+k}s_{n-k} = \alpha_k s_{n+1}s_{n-1} - \beta_k s_n^2 = W_k^2 s_{n+1}s_{n-1} - W_{k-1}W_{k+1}s_n^2$$

may be applied to $s_n = W_n$ yielding

$$W_{n+k}W_{n-k} = W_k^2 W_{n+1}W_{n-1} - W_{k-1}W_{k+1}W_n^2.$$

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General additional formula

$$s_{a+b}s_{a-b}w_{c+d}w_{c-d}+s_{a+d}s_{a-d}w_{b+c}w_{b-c}-s_{a+c}s_{a-c}w_{b+d}w_{b-d}=0$$

is an equivalent analogue of forbidden Weierstrass three term identity

$$\begin{gathered}\sigma(a+b)\sigma(a-b)\sigma(c+d)\sigma(c-d)-\\ -\sigma(a+c)\sigma(a-c)\sigma(b+d)\sigma(b-d)+\\ +\sigma(a+d)\sigma(a-d)\sigma(b+c)\sigma(b-c)=0.\end{gathered}$$

Equivalent form:

$$\mathrm{Pf}\begin{pmatrix} 0 & s_{k+l}w_{k-l} & s_{k+m}w_{k-m} & s_{k+n}w_{k-n} \\ s_{l+k}w_{l-k} & 0 & s_{l+m}w_{l-m} & s_{l+n}w_{l-n} \\ s_{m+k}w_{m-k} & s_{m+l}w_{m-l} & 0 & s_{m+n}w_{m-n} \\ s_{n+k}w_{n-k} & s_{n+l}w_{n-l} & s_{n+m}w_{n-m} & 0 \end{pmatrix}=0.$$

Conclusions

- Main properties of Somos-4 sequences can be drived by purely algebraic way without any help of special functions.
- First integral and elliptic divisibility sequences arise naturally on this way.
- Hopefully this approach will be suitable for higher rank sequences.

Properties which seemed to be more or less equivalent for general Somos sequences:

- Integrality
- Laurent phenomenon
- Finite rank
- Periodicity (mod N)
- General formula in terms of theta-functions

A natural candidate for experiments is the Gale – Robinson sequence generated by

$$s_{m+n}s_m = \alpha s_{m+r}s_{m+n-r} + \beta s_{m+p}s_{m+n-p} + \gamma s_{m+q}s_{m+n-q},$$

where $r + p + q = n$.

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Questions?