Discrepancy bounds for β-adic Halton sequences

J. M. Thuswaldner

Department of Mathematics and Information Technology
University of Leoben
Austria

CIRM Marseille, September 2018

Van der Corput and Halton sequences

- $q \in \mathbb{N}$, $q \geq 2$ (base).
- *q*-ary expansion: $n \in \mathbb{N}$,

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j \qquad (\varepsilon_j(n) \in \{0, \ldots, q-1\}).$$

- Van der Corput sequence (1935):

$$v_q(n) = \sum_{j=0}^{\infty} \varepsilon_j(n) q^{-j-1} \in [0,1) \qquad (n \geq 0).$$

- Halton sequence (1960): $s \geq 2$, $\boldsymbol{q} = (q_1, \ldots, q_s)$, $q_i \geq 2$,

$$h_{\boldsymbol{q}}(n) = (v_{q_1}(n), \ldots, v_{q_s}(n)) \in [0,1)^s \qquad (n \geq 0).$$

Discrepancy

Definition

- $s \in \mathbb{N}$, $A \subset [0,1)^s$
- $\mathbf{1}_A$ is the characteristic function of A
- $(\mathbf{y}_n)_{n\geq 0}$ a sequence in $[0,1)^s$

The (star) discrepancy is given by

$$D_N((\mathbf{y}_n)_{n\geq 0}) = \sup_{0<\omega_1,\dots,\omega_s\leq 1} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{[0,\omega_1)\times\cdots\times[0,\omega_s)}(\mathbf{y}_n) - \omega_1 \cdots \omega_s \right|.$$

- $D_N((\mathbf{y}_n)_{n\geq 0}) \to 0$ for $N \to \infty \iff (\mathbf{y}_n)_{n\geq 0}$ is equidistributed
- $D_N((v_q(n))_{n\geq 0}) \ll \log N/N$
- $D_N((h_{\mathbf{q}}(n))_{n\geq 0}) \ll (\log N)^s/N$

Linear recurrent number systems

- $m \geq 2$: m-bonacci sequence

$$\begin{aligned} F_k^{(m)} &= 2^k, & 0 \leq k \leq m-1, \\ F_k^{(m)} &= \sum_{j=1}^{m} F_{k-j}^{(m)}, & k \geq m. \end{aligned}$$

- Dominant root: $\beta = \varphi_m$ called m-bonacci number.
- $n \in \mathbb{N}$: the greedy expansion is given by

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) F_j^{(m)},$$

$\ldots \varepsilon_1(n)\varepsilon_0(n) \in \{0,1\}^{\mathbb{N}}$ with no block of m consecutive 1s.

β-adic versions

Definition

- β-adic van der Corput sequence

$$V_\beta(n) = \sum_{j\geq 0} \varepsilon_j(n)\beta^{-j-1}.$$

- $\boldsymbol{\beta}$-adic Halton sequence

$$H_{\boldsymbol{\beta}}(n) = (V_{\beta_1}(n), \ldots, V_{\beta_s}(n)) \qquad (\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)).$$

In our case $\beta = \varphi_m$ and $\beta_i = \varphi_{m_i}$.

- Ninomiya (1998): $D_N((V_\beta(n))_{n\geq 0}) \ll \log N/N$ (general β).
- Hofer, Iacò, and Tichy (2015): $H_{\boldsymbol{\beta}}(n)$ is equidistributed.
- Drmota (2015): Discrepancy results for "hybrid" Halton.

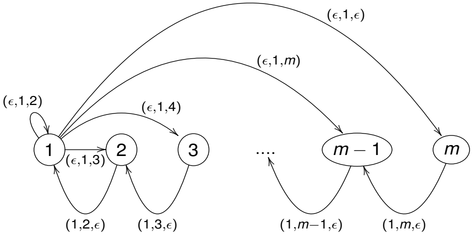
Substitutions associated with $(F_k^{(m)})$

$$\sigma_m(i) = \begin{cases} 1(i+1) & \text{for } i < m, \\ 1 & \text{for } i = m, \end{cases}$$

$B_m = B_{\sigma_m}$ incidence matrix: $(B_m)_{ij} = |\sigma(j)|_i$.

$$F_k^{(m)} = |\sigma_m^k(1)| \qquad (k \geq 0).$$

Prefix-suffix graph:



Rauzy fractals

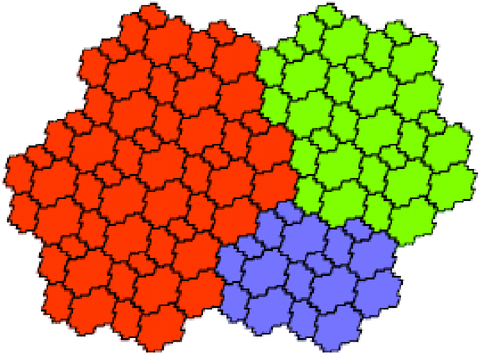
$$\mathcal{R}_m(i) = \bigcup_{i \xrightarrow{(p,i,s)} j} B_\sigma R(j)_m + \pi_c \mathbf{l}(p) \qquad (1 \leq i \leq m).$$

$$\mathcal{S}_k^{(m)} = \mathcal{S}_k = \Big\{ B_\sigma^k \mathcal{R}_m(i_k) + \pi_c \mathbf{l}\Big(\sigma_m^{k-1}(p_{k-1}) \dots \sigma_m(p_1) p_0\Big) \ : \ i_0 \xrightarrow{p_0} \cdots \xrightarrow{p_{k-1}} i_k \in G_{\sigma_m} \Big\}.$$

The elements of $\mathcal{S}_k$ overlap only on their boundaries,

$$\mathcal{R}_m = \bigcup_{S \in \mathcal{S}_k} S \quad \simeq \quad \text{fundamental domain of } \mathcal{L}_m.$$

The tribonacci Rauzy fractal and its subdivisions



The main result

Theorem (T. 2017)

Let $m_1, \ldots, m_s$ be pairwise distinct integers greater than or equal to 2 and set $\beta = (\varphi_{m_1}, \ldots, \varphi_{m_s})$. If

$$\{1, \varphi_{m_1}, \ldots, \varphi_{m_1}^{m_1-1}, \ldots, \varphi_{m_s}, \ldots, \varphi_{m_s}^{m_s-1}\}$$

is linearly independent over $\mathbb{Q}$ then the discrepancy of the β-adic Halton sequence $H_\beta(n)$ satisfies

$$D_N((H_\beta(n))_{n\geq 0}) \ll N^{\frac{\max\{d_i-(m_i-1) : 1\leq i\leq s\}}{(m_1-1)+\cdots+(m_s-1)}+\varepsilon}$$

for each $\varepsilon > 0$. Here $d_i = \dim_{\mathrm{B}}(\partial\mathcal{R}_{m_i})$, which is strictly smaller than $m_i - 1$, denotes the box counting dimension of the boundary of the Rauzy fractal $\mathcal{R}_{m_i}$, $1 \leq i \leq s$.

Van der Corput and rotations on the Rauzy fractal

- $n = \sum_{j\geq 0} \varepsilon_j(n) F_j^{(m)}$
- Choose $k \in \mathbb{N}$ arbitrary.
- $0 \leq r < m$: $\varepsilon_{k-r-1}(n) = 0$, $\varepsilon_{k-r}(n) = \cdots = \varepsilon_{k-1}(n) = 1$.

1. For $\mu_k = \sum_{j=0}^{k-1} \varepsilon_{k-1-j}(n)\varphi_m^j$ we have that

$$V_{\varphi_m}(n) \in \left[\frac{\mu_k}{\varphi_m^k}, \frac{\mu_k+\varphi_m^r-\sum_{i=0}^{r-1}\varphi_m^i}{\varphi_m^k}\right).$$

2. For $\nu_k = \sum_{j=0}^{k-1} \varepsilon_j(n) F_j^{(m)}$ we have that

$$n\pi_c(\mathbf{e}_1) \in \nu_k\pi_c(\mathbf{e}_1) + B_m^k \bigcup_{i=1}^{m-r} \mathcal{R}_m(i) \pmod{\mathcal{L}_m}.$$

3. The measures of the occurring sets agree, *i.e.*, we have

$$\left|\left[\frac{\mu_k}{\varphi_m^k}, \frac{\mu_k+\varphi_m^r-\sum_{i=0}^{r-1}\varphi_m^i}{\varphi_m^k}\right)\right| = \lambda_{\mathbf{v}}\left(\nu_k\pi_c(\mathbf{e}_1) + B_m^k \bigcup_{i=1}^{m-r} \mathcal{R}_m(i)\right).$$

The discrepancy of β-adic van der Corput

Lemma

Fix $m \geq 2$*, let* $N \in \mathbb{N}$ *be given, and choose* L *in a way that* $F_{L-1}^{(m)} \leq N - 1 < F_L^{(m)}$. *Then*

$$D_N((V_{\varphi_m}(n))_{n \geq 0}) \ll \frac{1}{\varphi_m^L} + \sum_{1 \leq k \leq L} \delta_k,$$

where

$$\delta_k = \sup_{S \in \mathcal{S}_k} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_S\Big(n\pi_c(\mathbf{e}_1) + \alpha_k \bmod \mathcal{L}_m\Big) - \lambda_{\mathbf{v}}(S) \right|$$

with $\alpha_k = \alpha_k(N)$ *chosen in a certain way.*

The discrepancy of β-adic Halton

Lemma

Fix $m_1, \dots, m_s \geq 2$, let $N \in \mathbb{N}$ be given, and choose $L_1, \dots, L_s$ in a way that $F_{L_j-1}^{(m_j)} \leq N-1 < F_{L_j}^{(m_j)}$. Then for $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s) = (\varphi_{m_1}, \dots, \varphi_{m_s})$

$$D_N((H_{\boldsymbol{\beta}}(n))_{n\geq 0}) \ll \sum_{i=1}^{s} \frac{1}{\beta_i^{L_i}} + \sum_{1\leq k_1 \leq L_1} \cdots \sum_{1\leq k_s\leq L_s} \delta_{k_1,\dots,k_s},$$

where

$\delta_{k_1,\dots,k_s} = \sup_{S_1\in\mathcal{S}_{k_1}^{(m_1)},\dots,S_s\in\mathcal{S}_{k_s}^{(m_s)}}$
$\left|\frac{1}{N}\sum_{n=0}^{N-1}\prod_{i=1}^{s}\mathbf{1}_{S_i}\left(n\pi_{c,m_i}(\mathbf{e}_1)+\alpha_{k_i} \bmod \mathcal{L}_{m_i}\right) - \prod_{i=1}^{s}\lambda_{\mathbf{v}_{m_i}}(S_i)\right|.$

Here $\alpha_{k_i} = \alpha_{k_i}(N)$ is chosen in a certain way.

Discrepancy of algebraic rotations

Lemma

Let $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{R}s$ *with algebraic numbers* $\gamma_1, \ldots, \gamma_s$ *and assume that* $\{1, \gamma_1, \ldots, \gamma_s\}$ *is linearly independent over* $\mathbb{Q}$*. Then for each* $\varepsilon > 0$ *we have*

$$D_N((n\gamma \bmod [0,1)^s)_{n \geq 0}) \ll N^{\varepsilon - 1}.$$

Proof.

Using a classical result by Wolfgang Schmidt (1970) we see that a vector $(\gamma_1, \ldots, \gamma_s)$ of real algebraic numbers for which $\{1, \gamma_1, \ldots, \gamma_s\}$ is linearly independent over $\mathbb{Q}$ is of finite type 1. Type 1 implies the stated discrepancy bound; see the book by Kuipers and Niederreiter (1974). □

The main result again

Theorem (T. 2017)

Let $m_1, \dots, m_s$ be pairwise distinct integers greater than or equal to 2 and set $\beta = (\varphi_{m_1}, \dots, \varphi_{m_s})$. If

$$\{1, \varphi_{m_1}, \dots, \varphi_{m_1}^{m_1-1}, \dots, \varphi_{m_s}, \dots, \varphi_{m_s}^{m_s-1}\}$$

is linearly independent over $\mathbb{Q}$ then the discrepancy of the β-adic Halton sequence $H_\beta(n)$ satisfies

$$D_N((H_\beta(n))_{n\geq 0}) \ll N^{\frac{\max\{d_i-(m_i-1):1\leq i\leq s\}}{(m_1-1)+\cdots+(m_s-1)}+\varepsilon}$$

for each $\varepsilon > 0$. Here $d_i = \dim_{\mathrm{B}}(\partial\mathcal{R}_{m_i})$, which is strictly smaller than $m_i - 1$, denotes the box counting dimension of the boundary of the Rauzy fractal $\mathcal{R}_{m_i}$, $1 \leq i \leq s$.

The easiest example

Example

Consider the golden mean φ_2 and the dominant root φ_3 of the *tribonacci polynomial* $X^3 - X^2 - X - 1$. The Rauzy fractal $\mathcal{R}_2$ is an interval, hence, $\dim_B \partial\mathcal{R}_2 = 0$. For $\mathcal{R}_3$ we know from Ito and Kimura (1991) that $\dim_B \partial\mathcal{R}_3 = 1.09336\ldots$ Since $\mathbb{Q}(\varphi_2, \varphi_3)$ has degree 6 over $\mathbb{Q}$ the linear independence assumption in the main theorem is satisfied and we obtain (choosing $\varepsilon > 0$ occurring in the theorem sufficiently small)

$$D_N((H_{(\varphi_2,\varphi_3)}(n))_{n\geq 0}) \ll N^{-0.30221}.$$

Future projects

1. Generalize to $G_{n+d} = a(G_{n+d-1} + \cdots + G_n)$ with $a \geq 2$
   - Characteristic roots are no longer units.
   - Rauzy fractals live in open subrings of adèle rings ($\mathfrak{p}$-adic factors).
   - $\mathfrak{p}$-adic version of approximation theorems (Schlickewei) needed.
2. More general recurrences
   - Language is no longer symmetric; see Ninomiya.
   - Extensions to Dumont-Thomas Numeration.
3. Bounded remainder sets; see Steiner (2006) for van der Corput.
4. Improvement of discrepancy estimates.