The sum of digits in two different bases

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Joint work with R. de la Bretèche and G. Tenenbaum

Work supported by the ANR–FWF project "MuDeRa".

Context – Sum of Digits

- It is generally conjectured that the base a and base b expansions of integers are statistically independent (provided a and b are multiplicatively independent).
- Also, it is natural to expect a phenomenon of dependancy for an infinity of exceptional integers.

For $a \geq 2$ and $n = \sum_{i\geq 0} \varepsilon_i a^i$ denote

$$s_a(n) = \sum_{i\geq 0} \varepsilon_i$$

the sum of digits in base a of the integer n.

Comparing the sum of digits in two different bases

Stewart (1980) (improving on Senge & Strauss (1970)) : If $\log a/\log b$ is irrational then

$$s_a(n) + s_b(n) > \frac{\log\log n}{\log\log\log n + C} - 1,$$

for all $n > 25$ and some effectively computable $C = C(a, b)$.

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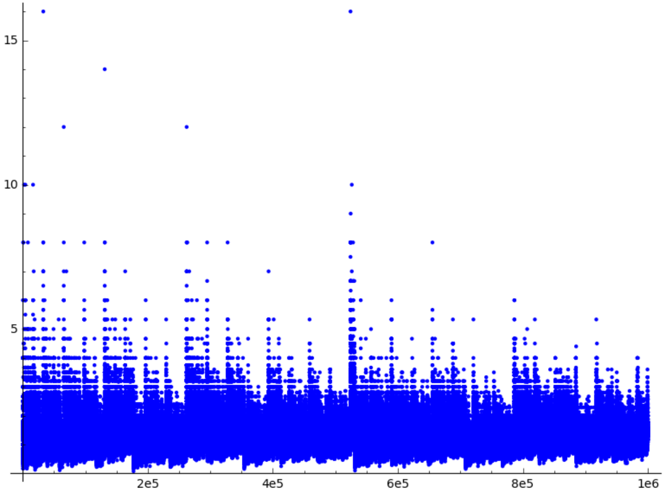
Question : Are there infinitely many n such that $s_3(n) = s_2(n)$?

$s_3(n)/s_2(n)$ for $n \leq 10^6$



s_b versus s_a

For almost all n, we have

$$s_b(n) \sim \frac{b-1}{2\log b}\log n, \qquad s_a(n) \sim \frac{a-1}{2\log a}\log n.$$

Let

$$\tau_0 = \tau_0(a,b) := \frac{(b-1)\log a}{(a-1)\log b}.$$

If $\tau = \tau_0$ then $s_b(n) \sim \tau s_a(n)$ evidently holds on a subset of $\mathbb{N}$ of density one.

For $s_3(n) \sim \tau_0 s_2(n)$ we have $\tau_0 = (\log 4)/\log 3 \approx 1.26186$ for a subset of density one.

Motivation

Theorem (Deshouillers, Habsieger, Landreau, Laishram (2017))

We have

$$\#\{n \leq x : |s_3(n) - s_2(n)| \leqslant 0.14572 \log n\} \gg x^{0.970359}.$$

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Remark : In the case where $a^k = b^\ell$ with k, ℓ some positive integers, the problem is trivial since it is sufficient to consider those n with digits 0 and 1 in base a^k for which we have $s_a(n) = s_b(n)$.

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Question : What about the accumulation points of $\{s_b(n)/s_a(n)\}_n$?

Main result

Theorem (R. de la Bretèche, TS, G. Tenenbaum (2018+))

Let $\tau > 0$, $a, b \geqslant 2$ mult. indep., and $c < c_0(a, b; \tau)$. There are $\gg x^c$ positive integers $n \leqslant x$ such that

$$s_b(n) \sim \tau s_a(n).$$

More precisely, if γ is an irrationality exponent of $(\log a)/\log b$, then

$$s_b(n) = \tau s_a(n)\left\{1 + O\left(\frac{1}{(\log n)^{\sigma/\gamma}}\right)\right\} \tag{1}$$

for all $\sigma \in]0, \Lambda/(6M^3 \log M)[$ for $M = \max(a, b)$ and some explicit $\Lambda = \Lambda(\tau_0; \tau)$.

Moreover, for all $\tau \neq \tau_0$, there is an exponent $d_0(\tau) = d_0(a, b; \tau) < 1$ such that (2) is realized by $\ll x^{d_0(\tau)+o(1)}$ positive integers $n \leqslant x$.

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For $s_3(n) \sim s_2(n)$ we have $c_0 \approx 0.94996$ and $d_0 \approx 0.993702$.

Irrationality exponent

Irrationality exponent : The proof only needs that $\vartheta := (\log a)/\log b$ has a finite irrationality exponent. This is always the case (Baker, 1972) if the quotient is irrational, i.e. there exists $\gamma \geqslant 2$ such that

$$\left|\vartheta - \frac{r}{q}\right| \gg \frac{1}{q^\gamma} \qquad \big(q \geqslant 1,\ (r,q) = 1\big).$$

Wu et Wang (2014) have shown that $\gamma = 5.117$ is admissible for $\vartheta := (\log 2)/\log 3$ (sharpening a result of Salikhov (2007)).

Binomial recentering

Let $\mathcal{M}_k$ be the set of integers $m \leqslant N_k := b^k - 1$ such that its base b expansion only contains the digits 0 and $b-1$. For $\varrho \in]0,1[$ we define a probability measure on $\mathcal{M}_k$ via

$$\mathbb{P}(m) = r_k(m) := \varrho^{s_b(m)/(b-1)}(1-\varrho)^{k-s_b(m)/(b-1)},$$

such that

$$\sum_{m\in\mathcal{M}_k} \mathbb{P}(m) = \sum_{0\leqslant j\leqslant k} \binom{k}{j} \varrho^j (1-\varrho)^{k-j} = 1.$$

We have

$$\mathbb{V}(s_b) = \sum_{m\in\mathcal{M}_k} r_k(m)\left\{s_b(m) - \varrho(b-1)k\right\}^2 = \varrho(1-\varrho)k(b-1)^2,$$

such that

$$\mathbb{P}\left(|s_b - \varrho(b-1)k| > T\sqrt{k}\right) \leqslant \frac{\varrho(1-\varrho)(b-1)^2}{T^2} \qquad (T \geqslant 1).$$

We show that the base a expansion of the elements of $\mathcal{M}_k$ are simply normal up to an exceptional set $\mathcal{E}_k$ with probability tending to 0 as $k \to \infty$. This implies

$$s_a(m) \sim \frac{a-1}{2\log a} \log N_k \qquad (m \in \mathcal{M}_k \setminus \mathcal{E}_k,\ k \to \infty).$$

This will imply (choosing appropriately $\rho = \rho(a, b; \tau)$)

$$\mathbb{P}(s_b \sim \tau s_a) = 1 + o(1), \qquad (k \to \infty).$$

Exponential sums

Let $\mathrm{e}(u) := \mathrm{e}^{2\pi i u}$ and set

$$\sigma_h(m,n) := \frac{1}{n} \sum_{1 \leqslant \nu \leqslant n} \mathrm{e}\left(\frac{hm}{a^\nu}\right) \qquad (m \in \mathcal{M}_k,\ n \geqslant 1, h \in \mathbb{Z})$$

and

$$\Delta_n(m) := \frac{1}{H+1} + \sum_{1 \leqslant h \leqslant H} \frac{|\sigma_h(m,n)|}{h}$$

Erdős-Turán inequality : For $I \subset [0,1]$ we have

$$\left| \frac{1}{n} \sum_{\substack{1 \leqslant \nu \leqslant n \\ \{m/a^\nu\} \in I}} 1 - |I| \right| \ll \Delta_n(m) \qquad (m \in \mathcal{M}_k,\ n \geqslant 1).$$

We choose $n := \lfloor \log N_k / \log a \rfloor$ and therefore have

$$s_a(m) = \frac{1}{2}(a-1)n + O\big(n\Delta_n(m)\big) = \frac{a-1}{2\log a}\log N_k + O\big(n\Delta_n(m)\big).$$

It remains to show that $\Delta_n(m) \to 0$ for $k \to \infty$ for almost all $m \in \mathcal{M}_k$.
For $H \geqslant 1$,

$$\mathbb{E}(\Delta_n) \leqslant \frac{1}{H+1} + \sum_{1 \leqslant h \leqslant H} \frac{1}{h}\mathbb{E}\big(|\sigma_h(m,n)|\big) \leqslant \frac{1}{H} + \sum_{1 \leqslant h \leqslant H} \frac{1}{h} M_h(n),$$

where

$$M_h(n)^2 = \frac{1}{n^2} \sum_{\mu,\nu \leqslant n} \mathrm{e}^{-8\varrho(1-\varrho)S_h(\mu,\nu)}$$

where

$$S_h(\mu,\nu) = \sum_{0 \leqslant j \leqslant k} \left\| h(b-1)b^j \left(\frac{1}{a^\nu} - \frac{1}{a^\mu}\right) \right\|^2.$$

We observe that

$$S_h(\mu,\nu) \geqslant S_h^*(\mu,\nu) := \sum_{L<j\leqslant \kappa+L} \|b^j\alpha\|^2, \qquad L = \lfloor \log h/\log b \rfloor$$

where

$$\alpha(h,\mu,\nu) := (b-1)b^{\{\frac{\log h}{\log b}\}-\{\nu\frac{\log a}{\log b}\}}\left(1-1/a^{\mu-\nu}\right).$$

Then (assuming $b \geq 3$), we have

$$S_h^*(\mu,\nu) \geq \frac{1}{b^2}\#\{\text{digits} = 1 \text{ in } \alpha \text{ with index in } [L+1, L+\kappa]\}.$$

A combinatorial inspection of the intervals of reals $\beta \in [0,(b-1)b]$ then shows that

$$S_h^*(\mu,\nu) \geq \frac{\kappa}{2b^3}$$

with the possible exception of $\{\nu\frac{\log a}{\log b}\}$ lying in a subset of $[0,1]$ of small measure. The number of exceptional integer ν can be bounded with the discrepancy of $(\{\nu\frac{\log a}{\log b}\})_{1\leq\nu\leq n}$.

This finally gives

$$M_h(n)^2 \ll \mathrm{e}^{-4\varrho(1-\varrho)\kappa/b^3} + \frac{1}{\sqrt{n}} + \mathrm{e}^{-\kappa/15b} + hb^\kappa \mathrm{e}^{-\kappa/15b} D_n,$$

where D_n is the discrepancy of $\{\nu\vartheta\}_{\nu=1}^n$ with $\vartheta = \log a/\log b$.

Now,

$$D_n \ll \frac{1}{q} + \frac{q}{n} \qquad \left(n \geqslant 1,\ |\vartheta - r/q| \leqslant 2/q^2\right)$$

and with the Dirichlet approximation theorem, we get $D_n \ll n^{-1/\gamma}$ where γ is an irrationality exponent of ϑ. It remains to choose κ accordingly to conclude.

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for all $\sigma \in]0, \Lambda/(6M^3 \log M)[$ for $M = \max(a, b)$ and some explicit $\Lambda = \Lambda(\tau_0; \tau)$.

Moreover, for all $\tau \neq \tau_0$, there is an exponent $d_0(\tau) = d_0(a, b; \tau) < 1$ such that (2) is realized by $\ll x^{d_0(\tau)+o(1)}$ positive integers $n \leqslant x$.

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Thank you !