The level of distribution of the Thue–Morse sequence

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Section 1

The Thue–Morse sequence

The Thue–Morse sequence

(1) Start with $\mathbf{t}^{(0)} = 0$ and let $\mathbf{t}^{(k+1)}$ be the concatenation of $\mathbf{t}^{(k)}$ and its Boolean complement $\overline{\mathbf{t}^{(k)}}$.

$$\begin{aligned}
\mathbf{t}^{(0)} &= 0\\
\mathbf{t}^{(1)} &= 01\\
\mathbf{t}^{(2)} &= 0110\\
\mathbf{t}^{(3)} &= 01101001\\
\mathbf{t}^{(4)} &= 0110100110010110\\
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\end{aligned}$$

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(2) By induction, it follows that $\mathbf{t}$ is the fixed point of the substitution

$$0 \mapsto 01, \quad 1 \mapsto 10$$

that starts with 0.

The Thue–Morse sequence, continued

(3) A third description uses the binary sum of digits function s:

$$s(\varepsilon_0 2^0 + \cdots + \varepsilon_\nu 2^\mu) = \varepsilon_0 + \cdots + \varepsilon_\nu \text{ for } \varepsilon_i \in \{0,1\}.$$

We have $\mathbf{t}_n = 0$ if and only if $s(n) \equiv 0 \bmod 2$.

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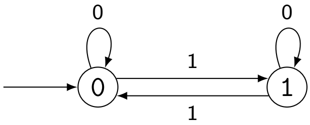
(4) The Thue–Morse sequence is one of the simplest automatic sequences:

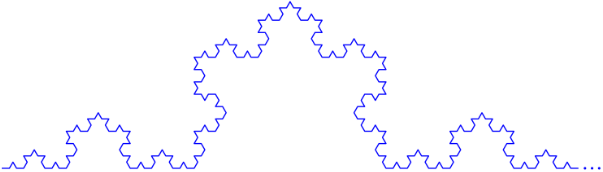


We feed in the binary expansion of n and obtain a letter $\in \{0, 1\}$.

The Thue–Morse sequence, continued

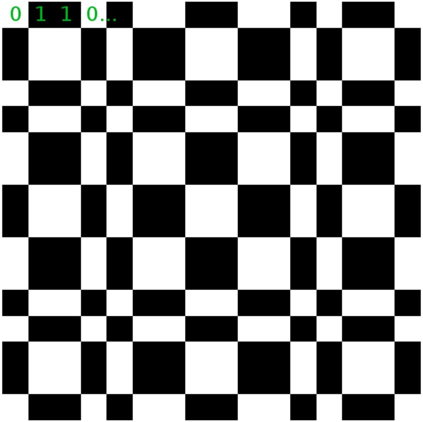
A less well-known characterization uses the Koch snowflake curve.

(5) The sequence $n \mapsto (-1)^{s(n)}\,\mathrm{e}(-n/3)$ describes the orientation of the nth line segment in the unscaled snowflake curve (where $\mathrm{e}(x) = e^{2\pi i x}$):



$\rightarrow$ The snowflake curve is the Thue–Morse sequence in disguise.

Thue–Morse, 16×16.



The Thue–Morse sequence has low *subword complexity*: if $p(L)$ denotes the number of (contiguous) subwords of length L, then $p(L) \le CL$ for some constant C. This is true for any automatic sequence. Here $C \le 8$, in fact $\limsup_{L\to\infty} p(L)/L = 10/3$.

Thue–Morse along arithmetic progressions

011010011001011010010110011010011001011001101001011010

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↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

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Theorem (Gelfond 1968)

Let $d \ge 1$ and a be integers. There is an absolute $\lambda < 1$ such that

$$\bigl|\{1 \le n \le x : \mathbf{t}_n = 0, n \equiv a \bmod d\}\bigr| = \frac{x}{2d} + \mathcal{O}\bigl(x^\lambda\bigr).$$

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That is, for all $d \geq 1$ and $a < d$ we have

$$\sum_{1 \leq m \leq M} (-1)^{s(md+a)} \leq CM^\lambda.$$

The Thue–Morse sequence shows *cancellation* along arithmetic progressions.

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Open problem: For k given, determine the subword complexity of $\mathbf{t}_{nk}$.

The error term for sums over APs

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  2. van der Waerden's theorem.
- Therefore we look at a certain average over d.

Section 2

The level of distribution

The averaged error term

Theorem (Fouvry–Mauduit 1996)

$$\sum_{1\le d\le D} \max_{\substack{y,z\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} (-1)^{s(n)} \right| \le C x^{1-\eta}$$

for some $\eta > 0$ *and* $D = x^{0.5924}$*.*

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Theorem (Müllner, S. 2017)

The Thue–Morse sequence has level of distribution $2/3$.

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Theorem (S. 2018+)

The Thue–Morse sequence has level of distribution 1*. More precisely, let* $0<\varepsilon<1$*. There exist* $\eta>0$ *and* C *such that*

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- This is a statement on *sparse* arithmetic progressions: the Thue–Morse sequence usually shows cancellation along N-term arithmetic progressions having common difference $\sim N^R$, where $R > 0$ is arbitrary ($R \le 1.46$ for Fouvry–Mauduit, $R < 2$ for Müllner–S.).

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Open problem. What about $D = x(\log x)^{-A}$? This corresponds to N-term APs with common difference $\sim e^{N^\varepsilon}$ for some $\varepsilon > 0$. Where is the limit?

Sparse arithmetic subsequences of $\mathbf{t}$

$\mathbf{t}$ along short arithmetic subsequences even seems to behave randomly. Such sequences even seem to pass most standard tests for PRNGs. Also, such a PRNG is reasonably fast on CPUs with POPCNT instruction.

Figure: $N = 64 \times 64$ terms, common difference $N^R = 3^{21}$

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Every finite sequence over $\{0, 1\}$ *appears as an arithmetic subsequence of the Thue–Morse sequence.*

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- However: for given d and a, there are always blocks that do not occur in $\mathbf{t}_{nd+a}$, since the subword complexity is at most linear! These sequences are therefore not *normal sequences*.
  $\rightsquigarrow$ consider sparse infinite subsequences of $\mathbf{t}$.

Section 3

Sparse infinite subsequences of Thue–Morse

Subsequences of $\mathbf{t}$

We wish to study $\mathbf{t}$ along subsequences of asymptotic density 0 and show (simple) normality of such subsequences. Candidates:

- Polynomials with values in $\mathbb{N}$ } Gelfond problems
- Prime numbers } Gelfond problems
- 3^n
- $\lfloor f(n) \rfloor$ for f satisfying some growth conditions. For example, Piatetski-Shapiro sequences $\lfloor n^c \rfloor$.

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Theorem (Special case of Mauduit–Rivat 2009, Acta Math.)

The Thue–Morse sequence along the sequence of squares is simply normal.

Notorious open problem: $\mathbf{t}_{n^3}$.

Theorem (Special case of Mauduit–Rivat 2010, Ann. of Math.)

The Thue–Morse sequence along the primes is simply normal.

Subsequences of $\mathbf{t}$

The sum of digits of $\lfloor n^c \rfloor$ is an approximation to the problem "the sum of digits of n^2", which could not be handled at first.

Theorem (Special case of Mauduit–Rivat 2005)

Let $1 < c < 1.4$*. There exists an* $\eta > 0$ *such that*

$$\sum_{1 \le n \le x} (-1)^{s(\lfloor n^c \rfloor)} \ll x^{1-\eta}.$$

In particular, for $1 < c < 1.4$, the sequence $n \mapsto \mathbf{t}_{\lfloor n^c \rfloor}$ is simply normal.

The Thue–Morse sequence along sparse subsequences

Theorem (S. 2014)

The Thue–Morse sequence along $\lfloor n^c \rfloor$ *is simply normal for* $1 < c \leq 1.42$.

Note that 1.42 is larger than 1.4 by “two cents”.

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Theorem (Drmota–Mauduit–Rivat 2018)

The Thue–Morse sequence along the sequence of squares is normal: every block $B \in \{0,1\}^k$ *appears as a subword with asymptotic frequency* $1/2^k$.

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Theorem (Müllner–S. 2017)

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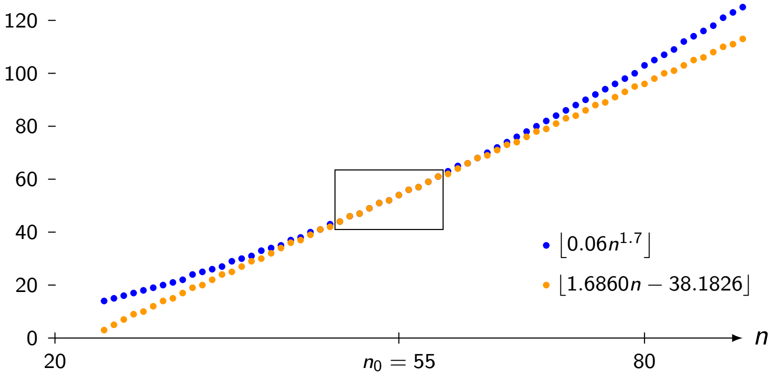
Piatetski-Shapiro via Beatty sequences

The question on $\lfloor n^c \rfloor$ can be reduced to the linear case by Taylor approximation. This simplification comes at the cost of shorter sums.



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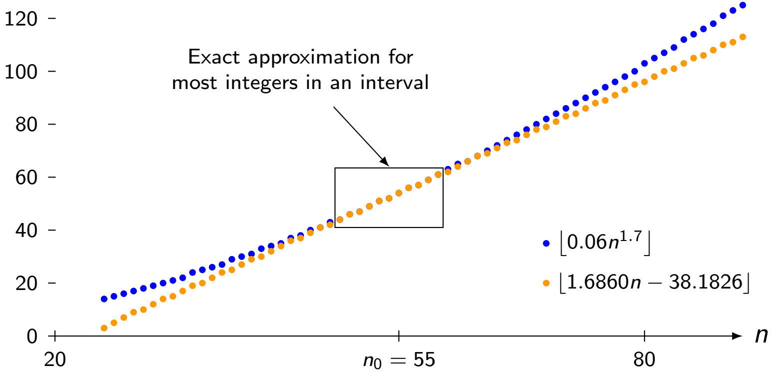

For $c \to 2$, the slope $f'(n)$ is a large power of the length of the approximation interval.

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Proposition (S. 2014)

We write $f(x) = x^c$, where $1 < c < 2$ is a real number. There exists a constant C such that for all $N \ge 2$ and $K > 0$ we have

$$\left| \frac{1}{N} \sum_{N < n \le 2N} (-1)^{s(\lfloor n^c \rfloor)} \right| \le C \left(f''(N) K^2 + \frac{(\log N)^2}{K} + \frac{J(f'(N), K)}{f'(N) K} \right),$$

where

$$J(D, K) = \int_D^{2D} \max_{\beta \ge 0} \left| \sum_{0 \le n < K} (-1)^{s(\lfloor n\alpha + \beta \rfloor)} \right| \mathrm{d}\alpha.$$

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- The problem "simple normality" is reduced to a Beatty sequence version of our main theorem!
- The proof of this new statement is analogous to the main theorem.

Section 4

“Proof” of the main theorem

van der Corput's inequality

In the proof of the main theorem we make use of van der Corput's inequality:

Lemma

Let I be a finite interval containing N integers and let a_n be a complex number for $n \in I$. For all integers $K \geq 1$ and $R \geq 1$ we have

$$\left|\sum_{n\in I} a_n\right|^2 \leq \frac{N+K(R-1)}{R} \sum_{|r|<R} \left(1-\frac{|r|}{R}\right) \sum_{\substack{n\in I\\ n+Kr\in I}} a_{n+Kr}\overline{a_n}.$$

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Instead of the original sum, we now have to estimate certain correlations (where KR will be small compared to N).

Reducing the number of significant digits

- Iterating van der Corput's inequality, we introduce "multiple correlations", while reducing step by step the number of digits that have to be considered.

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$$\sum_{r_1,\ldots,r_m<2^\rho} \sum_{0\le n<2^\rho} \mathrm{e}\left(\frac{1}{2} \sum_{\varepsilon_1,\ldots,\varepsilon_m\in\{0,1\}} s_\rho\left(n + \sum_{1\le i\le m} \varepsilon_i r_i\right)\right)$$

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- This is a *Gowers norm* for the Thue–Morse sequence. An estimate of a very similar expression was given by Konieczny (2017). This finishes our "proof".

Thank you! [1]

[1] Supported by the FWF–ANR joint project MuDeRa