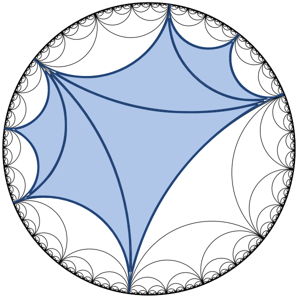
The Farey graph, continued fractions and SL_2-tilings

Ian Short



Thursday 4 October 2018

The Open University

Work in preparation with Peter Jørgensen (Newcastle)

Coxeter's friezes

Coxeter's friezes

$$
\begin{array}{cccccccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & \\
 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & \\
\cdots & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & \cdots \\
 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's friezes

$$
\begin{array}{cccccccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & \\
\cdots & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & \cdots \\
 & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & \\
 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

$$
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
\qquad\qquad ad - bc = 1
$$

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's friezes

$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
$$

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's friezes

$$\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}$$

Theorem (Coxeter) Every infinite strip of positive integers bordered by 0s that satisfies the unimodular rule $ad - bc = 1$ is periodic.

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's friezes

$$\begin{array}{ccccccccccccc} & 0 & & 0 & & 0 & & 0 & & 0 & & & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ \cdots & 2 & & 1 & & 2 & & 1 & & 2 & & & \cdots \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & & 0 & & & \end{array} \qquad \begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

Theorem (Coxeter) Every infinite strip of positive integers bordered by 0s that satisfies the unimodular rule $ad - bc = 1$ is periodic.

Definition An infinite strip of integers of this type is called a *positive integer frieze*.

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's friezes

$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
$$

Theorem (Coxeter) Every infinite strip of positive integers bordered by 0s that satisfies the unimodular rule $ad - bc = 1$ is periodic.

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Theorem (Coxeter) Every positive integer frieze is invariant under a glide reflection.

H.S.M. Coxeter, *Frieze patterns*, 1971

Integer friezes

$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & -1 & & -1 & & -1 & & -1 & & -1 & & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & & \cdots \\
 & & -1 & & -1 & & -1 & & -1 & & -1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$

Integer friezes

$$
\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & -1 & & -1 & & -1 & & -1 & & -1 & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots \\
 & & -1 & & -1 & & -1 & & -1 & & -1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & a & & 0 & & b & & 0 & & \\
\cdots & & -1 & & -1 & & -1 & & -1 & & -1 & \cdots \\
 & c & & 0 & & d & & 0 & & e & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Coxeter's question

$$\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}$$

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's question

$$\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}$$

Theorem Each positive integer frieze is specified by a finite sequence of positive integers.

H.S.M. Coxeter, *Frieze patterns*, 1971

Coxeter's question

$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
$$

Theorem Each positive integer frieze is specified by a finite sequence of positive integers.

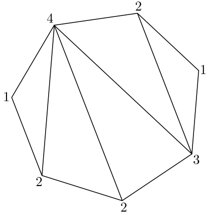
Question (Coxeter) Characterise positive integer friezes!

H.S.M. Coxeter, *Frieze patterns*, 1971

Conway's insight

	0		0		0		0		0		0		0		0		0		0		0		0		
		1		1		1		1		1		1		1		1		1		1		1		1	
	1		2		2		3		1		2		4		1		2		2		3		1		
		1		3		5		2		1		7		3		1		3		5		2		1	
…	2		1		7		3		1		3		5		2		1		7		3		1		…
		1		2		4		1		2		2		3		1		2		4		1		2	
	1		1		1		1		1		1		1		1		1		1		1		1		
		0		0		0		0		0		0		0		0		0		0		0		0	

width $= 7$
period $= 7$

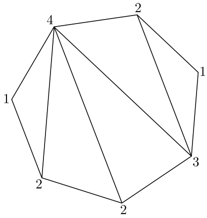


J.H. Conway & H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, 1973

Conway's insight

$$
\begin{array}{ccccccccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & & \\
\cdots & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & \cdots \\
 & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & & \\
 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

width $= 7$
period $= 7$



Theorem (Conway & Coxeter) There is a one-to-one correspondence between positive integer friezes of period n and triangulated n-gons.

J.H. Conway & H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, 1973

Counting friezes

Theorem There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ positive integer friezes of width n.

a. Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors:

b. Binary parenthesizations of a string of $n+1$ letters:

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

⋮

mmm. Positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (necessarily with $n+1$ rows)

$$\begin{array}{cccccccccccccc}
1 & 1 & 1 & \cdots & 1 & & 1 & & 1 & \cdots & 1 & & 1 \\
 & a_1 & a_2 & a_3 & \cdots & a_{n+2} & & a_1 & & a_2 & \cdots & a_{n-1} & \\
 & & b_1 & b_2 & b_3 & \cdots & b_{n+2} & b_1 & & \cdots & b_{n-2} & & \\
 & & & & & \vdots & & & & & & & \\
 & & & r_1 & r_2 & r_3 & \cdots & r_{n+2} & r_1 & & & & \\
 & & & & 1 & 1 & 1 & \cdots & 1 & & & &
\end{array} \qquad (6.54)$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru + 1$ (an example of such an array for $(a_1, \ldots, a_8) = (1, 3, 2, 1, 5, 1, 2, 3)$

R.P. Stanley, Enumerative Combinatorics, vol. 2, 1999

SL_2-tilings

SL_2-tilings

$$
\begin{array}{cccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & \\
& 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 1 & & 2 & & 1 & & 2 & \\
& 1 & & 1 & & 1 & & 1 & & 1 \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
\xrightarrow{\text{rotate } 45^\circ}
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 0 & & & \\
& 0 & 1 & 2 & 1 & 0 & & \\
& & 0 & 1 & 1 & 1 & 0 & \\
& & & 0 & 1 & 2 & 1 & 0
\end{array}
$$

SL_2-tilings

$$\begin{array}{ccccccccccc} 0 & & 0 & & 0 & & 0 & & 0 & & \\ & 1 & & 1 & & 1 & & 1 & & 1 \\ 2 & & 1 & & 2 & & 1 & & 2 & & \\ & 1 & & 1 & & 1 & & 1 & & 1 \\ 0 & & 0 & & 0 & & 0 & & 0 & & \end{array} \xrightarrow{\text{rotate } 45^\circ} \begin{array}{cccccccc} 0 & 1 & 1 & 1 & 0 & & & \\ & 0 & 1 & 2 & 1 & 0 & & \\ & & 0 & 1 & 1 & 1 & 0 & \\ & & & 0 & 1 & 2 & 1 & 0 \end{array}$$

$$\begin{array}{c} \vdots \\ \cdots \begin{array}{cccccccccccccccc} 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & & & \\ & 0 & 1 & 2 & 1 & 0 & -1 & -2 & -1 & 0 & 1 & 2 & 1 & 0 & & \\ & & 0 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & \\ & & & 0 & 1 & 2 & 1 & 0 & -1 & -2 & -1 & 0 & 1 & 2 & 1 & 0 \end{array} \cdots \\ \vdots \end{array}$$

SL_2-tilings

Definition $\mathsf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$

SL_2-tilings

Definition $\mathsf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$

Definition An SL_2-*tiling* is an infinite array of integers such that any two-by-two submatrix satisfies $ad - bc = 1$.

$$
\begin{array}{ccccccc}
 & & & \vdots & & & \\
 & 5 & 9 & 4 & 7 & 17 & \\
 & 1 & 2 & 1 & 2 & 5 & \\
\cdots & 2 & 5 & 3 & 7 & 18 & \cdots \\
 & 1 & 3 & 2 & 5 & 13 & \\
 & 3 & 10 & 7 & 18 & 47 & \\
 & & & \vdots & & &
\end{array}
$$

SL_2-tilings

Definition $\mathsf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$

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$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 5 & 9 & 4 & 7 & 17 & \\ & 1 & 2 & 1 & 2 & 5 & \\ \cdots & 2 & 5 & 3 & 7 & 18 & \cdots \\ & 1 & 3 & 2 & 5 & 13 & \\ & 3 & 10 & 7 & 18 & 47 & \\ & & & \vdots & & & \end{array} \qquad \begin{array}{ccccccc} & & & \vdots & & & \\ & 13 & 8 & 3 & 4 & 5 & \\ & 8 & 5 & 2 & 3 & 4 & \\ \cdots & 3 & 2 & 1 & 2 & 3 & \cdots \\ & 4 & 3 & 2 & 5 & 8 & \\ & 5 & 4 & 3 & 8 & 13 & \\ & & & \vdots & & & \end{array}$$

A selection of results on SL_2-tilings

Classification of *tame* SL_2-tilings (Bergeron, Reutenauer)

F. Bergeron & C. Reutenauer, *SL_k-tilings of the plane*, 2010
C. Bessenrodt, P. Jørgensen & T. Holm, *All SL_2-tilings come from infinite triangulations*, 2017
K. Baur, M.J. Parsons & M. Tschabold, *Infinite friezes*, 2016

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Classification of *infinite friezes* (Baur, Parsons, Tschabold)

$$
\begin{array}{cccccccccc}
 & & & & \vdots & & & & & \\
 & 4 & 3 & 2 & 1 & 0 & & & & \\
 & & 4 & 3 & 2 & 1 & 0 & & & \\
\cdots & & & 4 & 3 & 2 & 1 & 0 & & \cdots \\
 & & & & 4 & 3 & 2 & 1 & 0 & \\
 & & & & \vdots & & & & &
\end{array}
$$

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Classification of *infinite friezes* (Baur, Parsons, Tschabold)

$$\cdots \quad \begin{array}{ccccccccc} & & & \vdots & & & & & \\ 4 & 3 & 2 & 1 & 0 & & & & \\ & 4 & 3 & 2 & 1 & 0 & & & \\ & & 4 & 3 & 2 & 1 & 0 & & \\ & & & 4 & 3 & 2 & 1 & 0 & \\ & & & \vdots & & & & & \end{array} \quad \cdots$$

And much more!

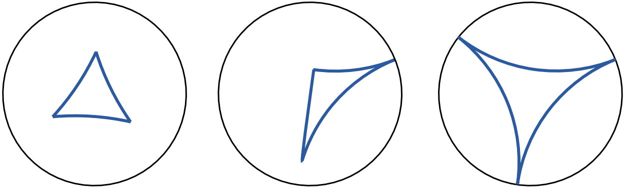
F. Bergeron & C. Reutenauer, *SL_k-tilings of the plane*, 2010

C. Bessenrodt, P. Jørgensen & T. Holm, *All SL_2-tilings come from infinite triangulations*, 2017

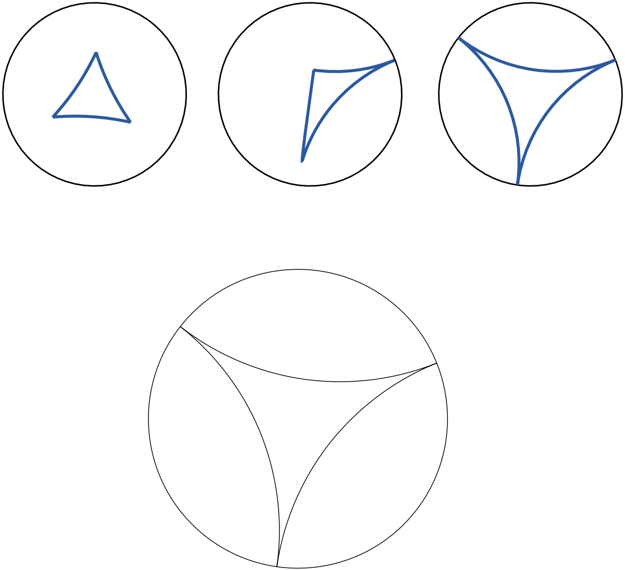
K. Baur, M.J. Parsons & M. Tschabold, *Infinite friezes*, 2016

The Farey graph

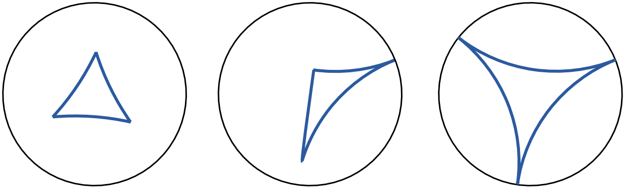
Triangles

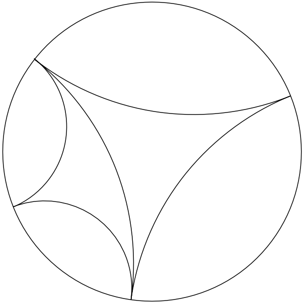


Triangles

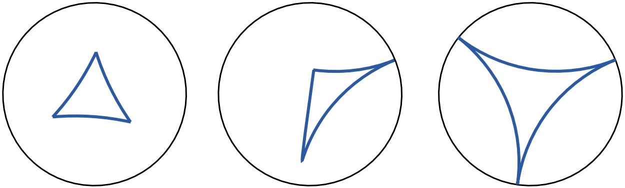


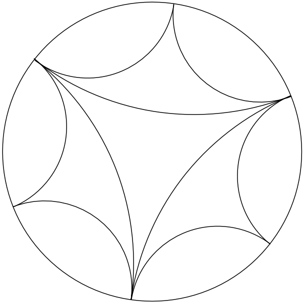
Triangles



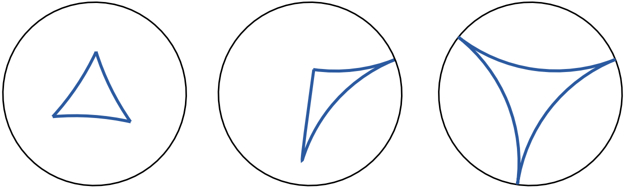


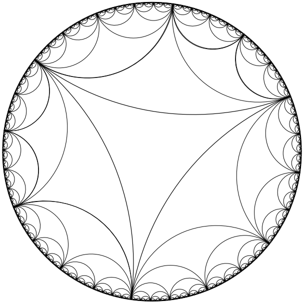
Triangles



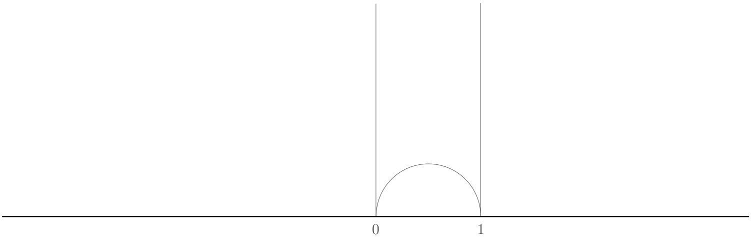


Triangles

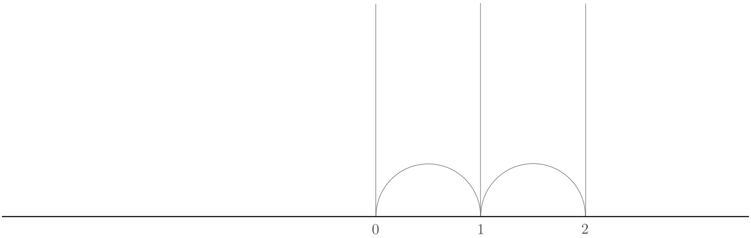




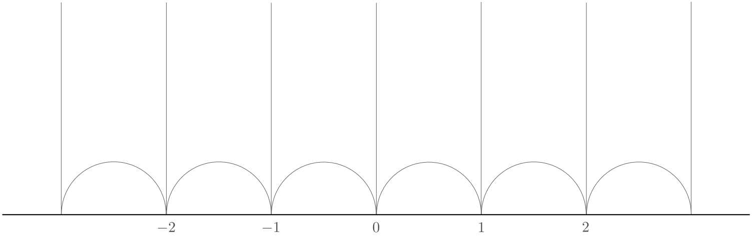
Farey graph



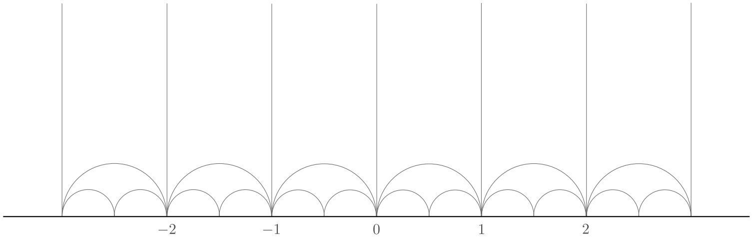
Farey graph



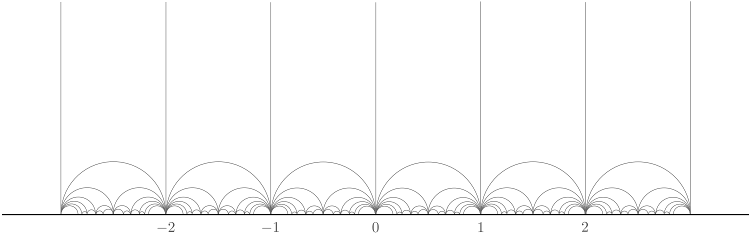
Farey graph



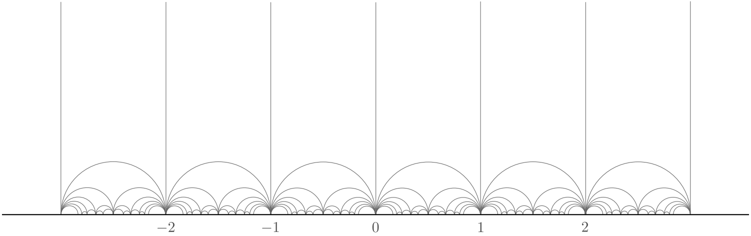
Farey graph



Farey graph

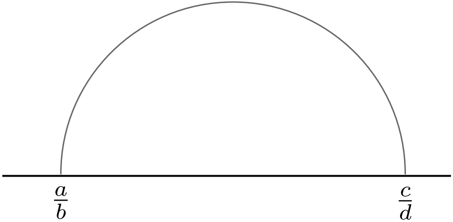


Farey graph

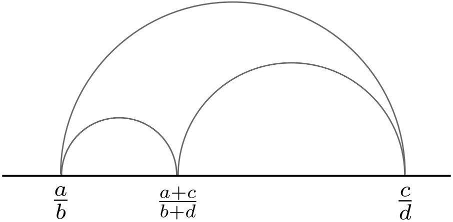


Definition The *Farey graph* is the graph with vertices $\mathbb{Q} \cup \{\infty\}$, and with an edge (represented by a hyperbolic line) from a/b to c/d if and only if $|ad - bc| = 1$.

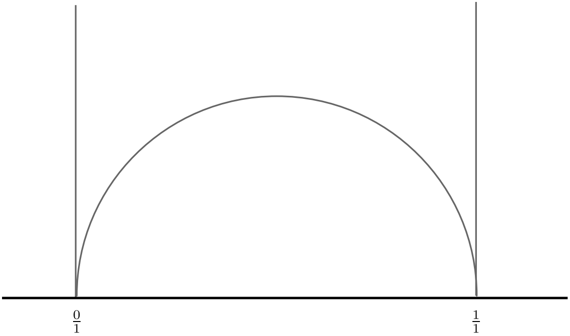
Farey addition



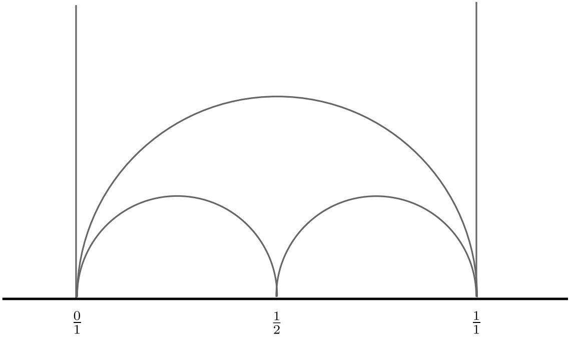
Farey addition



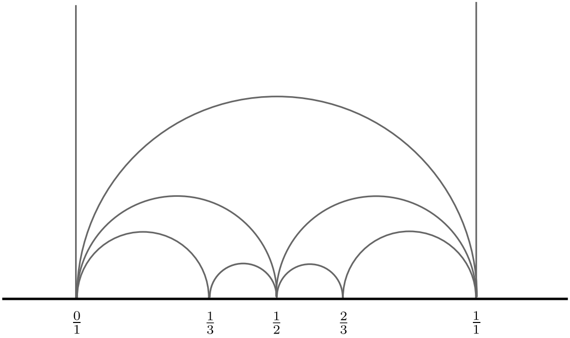
Farey sequences



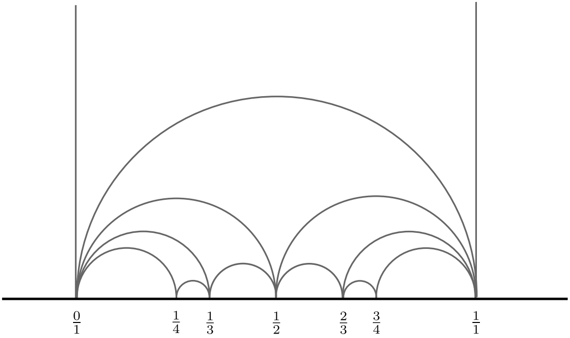
Farey sequences



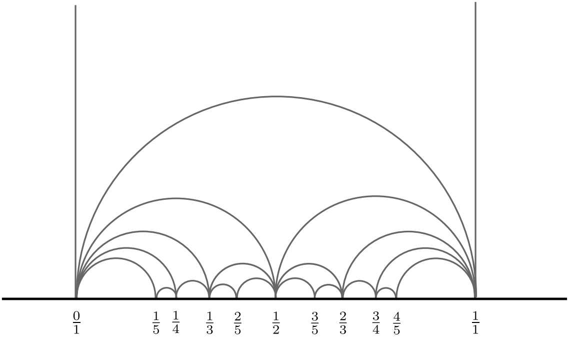
Farey sequences



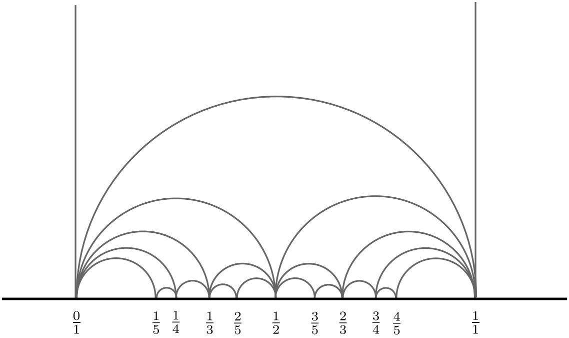
Farey sequences



Farey sequences



Farey sequences

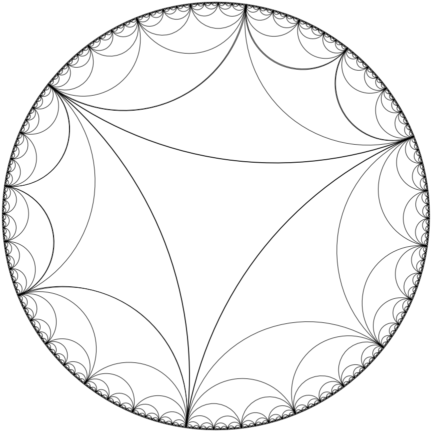


Farey's observation The *Farey sequence* of level n,

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}$$

(for $n = 6$), has the property that term k is the Farey sum of term $k - 1$ and term $k + 1$.

Automorphisms of the Farey graph



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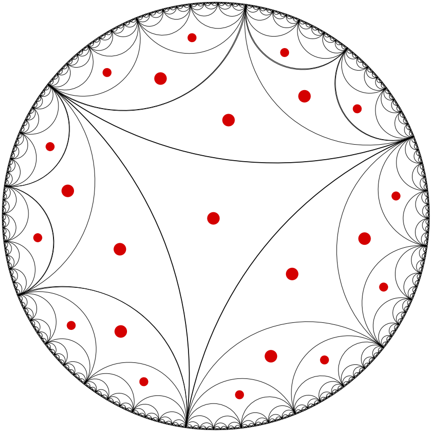

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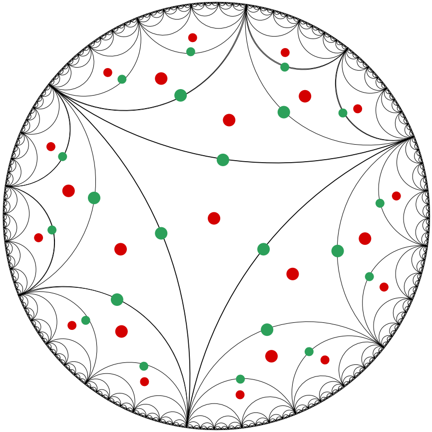
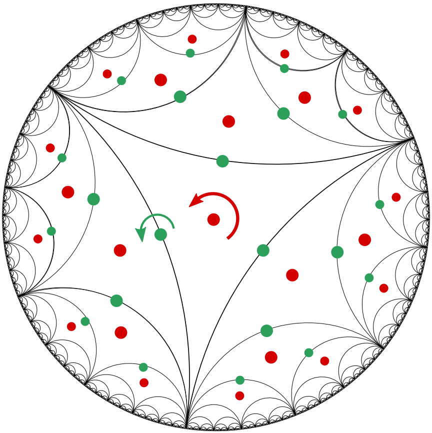


Automorphisms of the Farey graph



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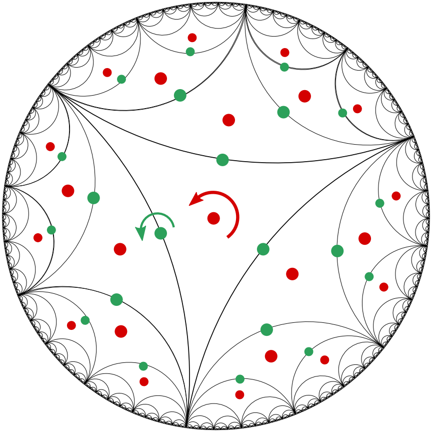
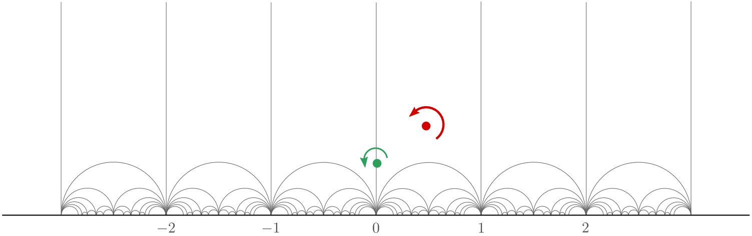
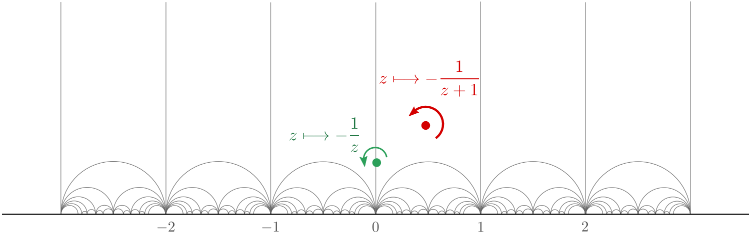


Automorphism group $\cong C_2 * C_3$

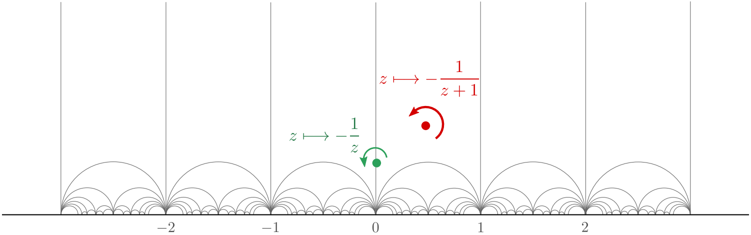
Modular group



Modular group



Modular group

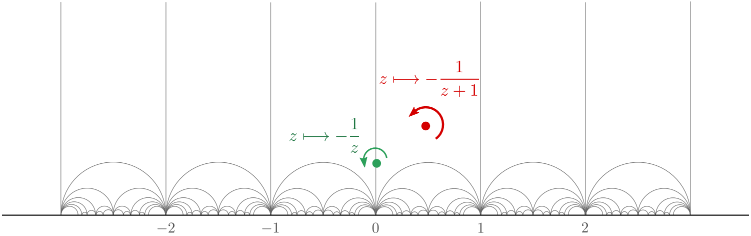


Definition The *modular group* is the group

$$\Gamma = \left\{ z \longmapsto \frac{az+b}{cz+d} \;:\; a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\},$$

which is generated by $-1/(1+z)$ and $-1/z$.

Modular group



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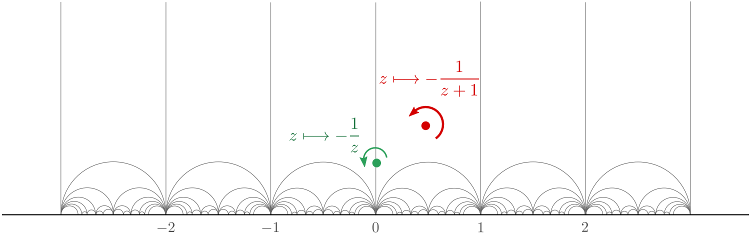

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Key property   The modular group is the group of automorphisms of the Farey graph.

Integer continued fractions

Euclid's algorithm

$$\frac{31}{13}$$

Euclid's algorithm

$$\frac{31}{13} \quad = \quad 2 + \frac{5}{13}$$

Euclid's algorithm

$$\frac{31}{13} = 2 + \frac{5}{13}$$
$$= 2 + \frac{1}{\frac{13}{5}}$$

Euclid's algorithm

$$\begin{aligned}\frac{31}{13} &= 2+\frac{5}{13}\\ &= 2+\frac{1}{\frac{13}{5}}\\ &= 2+\frac{1}{2+\frac{3}{5}}\end{aligned}$$

Euclid's algorithm

$$
\begin{aligned}
\frac{31}{13} &= 2 + \frac{5}{13} \\
&= 2 + \frac{1}{\frac{13}{5}} \\
&= 2 + \cfrac{1}{2 + \frac{1}{\frac{5}{3}}}
\end{aligned}
$$

Euclid's algorithm

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&= 2 + \frac{1}{\frac{13}{5}} \\
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&= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{2}{3}}}
\end{aligned}$$

Euclid's algorithm

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\frac{31}{13} &= 2 + \frac{5}{13} \\
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&= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{\frac{3}{2}}}}
\end{aligned}$$

Euclid's algorithm

$$\frac{31}{13} \quad = \quad 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

The nearest-integer algorithm

$$\frac{31}{13}$$

The nearest-integer algorithm

$$\frac{31}{13} \quad = \quad 2 + \frac{1}{\frac{13}{5}}$$

The nearest-integer algorithm

$$\frac{31}{13} = 2 + \frac{1}{\frac{13}{5}}$$

$$= 2 + \frac{1}{3 - \frac{2}{5}}$$

The nearest-integer algorithm

$$\begin{aligned}\frac{31}{13} &= 2+\frac{1}{\frac{13}{5}} \\ &= 2+\frac{1}{3+\frac{1}{-\frac{5}{2}}}\end{aligned}$$

The nearest-integer algorithm

$$\begin{aligned}\frac{31}{13} &= 2 + \cfrac{1}{\frac{13}{5}} \\ &= 2 + \cfrac{1}{3 + \cfrac{1}{-\frac{5}{2}}} \\ &= 2 + \cfrac{1}{3 + \cfrac{1}{-3 + \cfrac{1}{2}}}\end{aligned}$$

Another expansion

$$\frac{31}{13} = 3 + \cfrac{1}{-2 + \cfrac{1}{3 + \cfrac{1}{-3}}}$$

Another expansion

$$\frac{31}{13} \quad = \quad 3 + \cfrac{1}{-2 + \cfrac{1}{3 + \cfrac{1}{-3}}}$$

Question How many integer continued fraction expansions of $31/13$ are there?

Continued fraction approximants

Approximants

$$\frac{A_n}{B_n} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots + \cfrac{1}{b_n}}}}$$

Continued fraction approximants

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Calculating approximants

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}$$

Continued fraction approximants

Approximants

$$\frac{A_n}{B_n} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots + \cfrac{1}{b_n}}}}$$

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$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

Continued fraction approximants

Approximants

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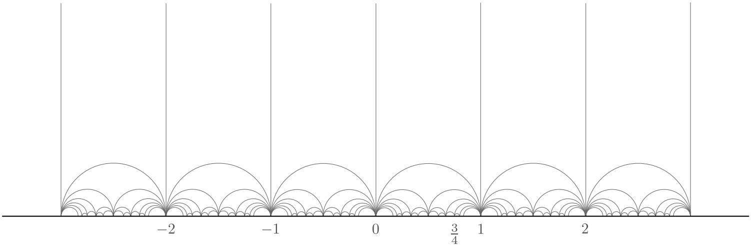
Calculating approximants

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}$$

$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

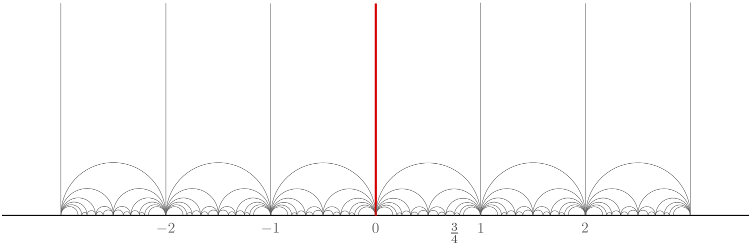
Crucial observation The approximants A_{n-1}/B_{n-1} and A_n/B_n are adjacent in the Farey graph.

Paths in the Farey graph



$$\frac{3}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}$$

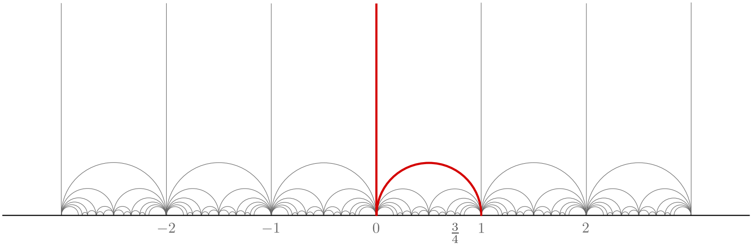
Paths in the Farey graph



$$\frac{3}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}$$

$$\frac{A_1}{B_1} = 0,$$

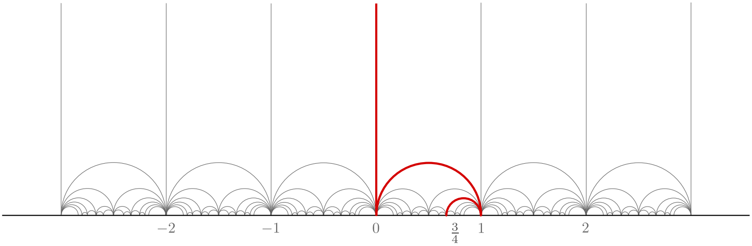
Paths in the Farey graph



$$\frac{3}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}$$

$$\frac{A_1}{B_1} = 0, \quad \frac{A_2}{B_2} = \frac{1}{1} = 1,$$

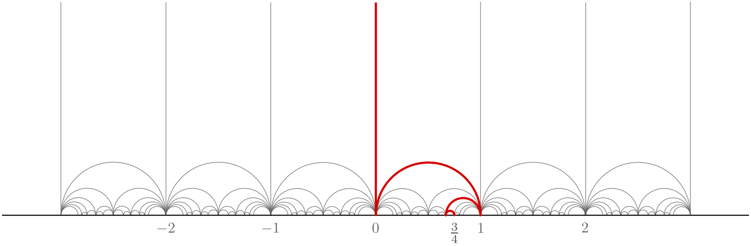
Paths in the Farey graph



$$\frac{3}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}$$

$$\frac{A_1}{B_1} = 0, \quad \frac{A_2}{B_2} = \frac{1}{1} = 1, \quad \frac{A_3}{B_3} = \cfrac{1}{1 + \cfrac{1}{2}} = \frac{2}{3},$$

Paths in the Farey graph



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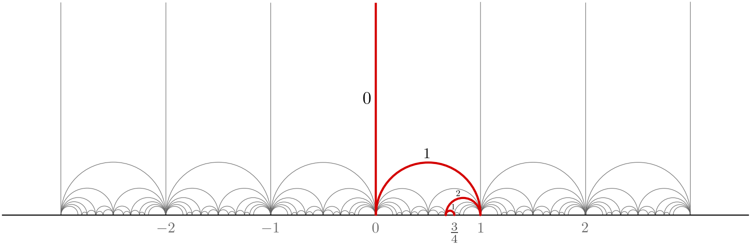
Paths in the Farey graph



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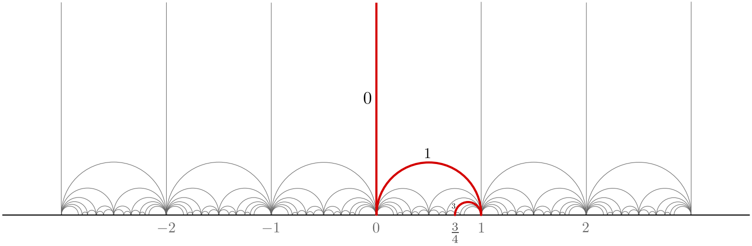
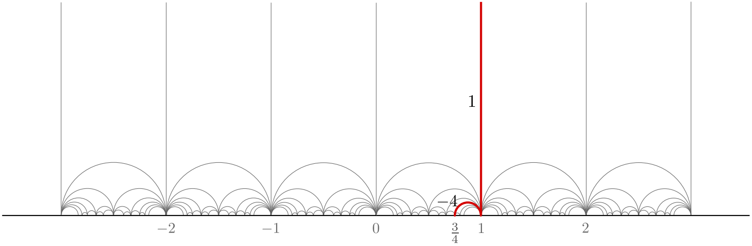
Paths in the Farey graph



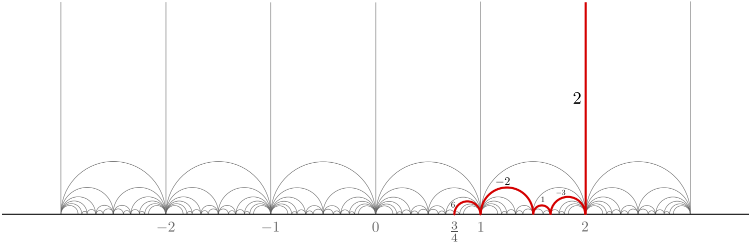
$$\frac{3}{4} = \cfrac{1}{1 + \cfrac{1}{3}}$$

Paths in the Farey graph



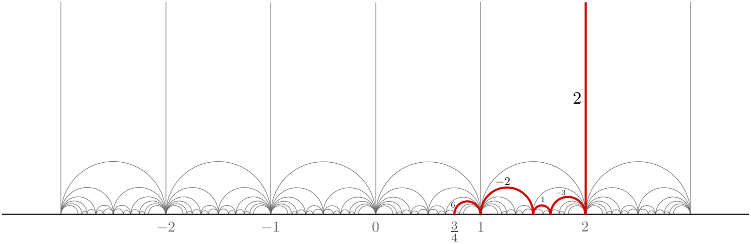
$$\frac{3}{4} = 1 + \frac{1}{-4}$$

Paths in the Farey graph



$$\frac{3}{4} = 2 + \cfrac{1}{-3 + \cfrac{1}{1 + \cfrac{1}{-2 + \cfrac{1}{6}}}}$$

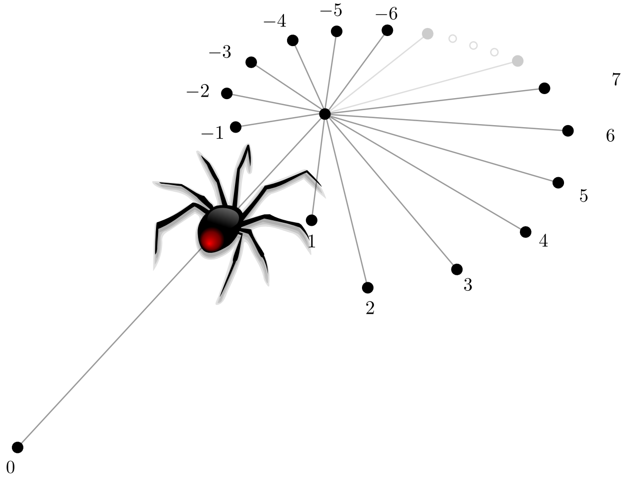
Paths in the Farey graph



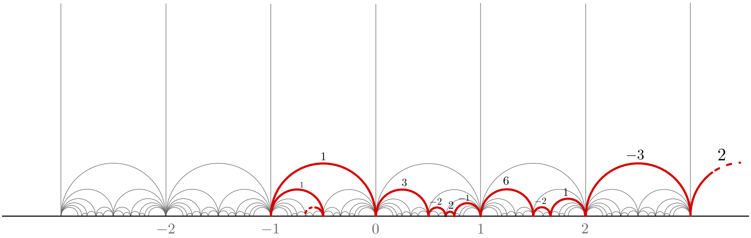
$$\frac{3}{4} = 2 + \cfrac{1}{-3 + \cfrac{1}{1 + \cfrac{1}{-2 + \cfrac{1}{6}}}}$$

Theorem There is a one-to-one correspondence between integer continued fractions and paths starting from ∞ in the Farey graph.

Navigating the Farey graph



Biinfinite continued fractions



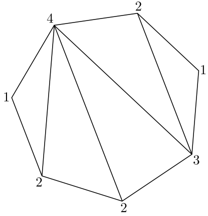
$$[\ldots, 1, 1, 3, -2, 2, -1, 6, -2, 1, -3, 2, \ldots]$$

Classifying integer tilings using the Farey graph

Conway's insight

0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 1
1 2 2 3 1 2 4 1 2 2 3 1
1 3 5 2 1 7 3 1 3 5 2 1
… 2 1 7 3 1 3 5 2 1 7 3 1 …
1 2 4 1 2 2 3 1 2 4 1 2
1 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0

width $= 7$
period $= 7$

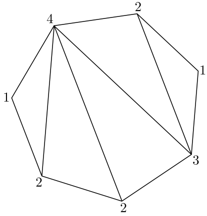


J.H. Conway & H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, 1973

Conway's insight

	0		0		0		0		0		0		0		0		0		0		0		0		
		1		1		1		1		1		1		1		1		1		1		1		1	
	1		2		2		3		1		2		4		1		2		2		3		1		
		1		3		5		2		1		7		3		1		3		5		2		1	
…	2		1		7		3		1		3		5		2		1		7		3		1		…
		1		2		4		1		2		2		3		1		2		4		1		2	
	1		1		1		1		1		1		1		1		1		1		1		1		
		0		0		0		0		0		0		0		0		0		0		0		0	

width $= 7$
period $= 7$



Theorem (Conway & Coxeter) There is a one-to-one correspondence between positive integer friezes of period n and triangulated n-gons.

J.H. Conway & H.S.M. Coxeter, *Triangulated polygons and frieze patterns*, 1973

Triangulated polygons in the Farey graph

Key observation Any triangulated polygon can be embedded in the Farey graph in essentially one way.

S. Morier-Genoud, V. Ovsienko & S. Tabachnikov, $\mathrm{SL}_2(\mathbb{Z})$*-tilings of the torus, Coxeter–Conway friezes and Farey triangulations*, 2015

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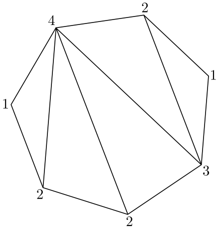
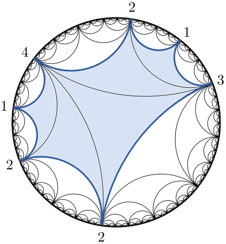
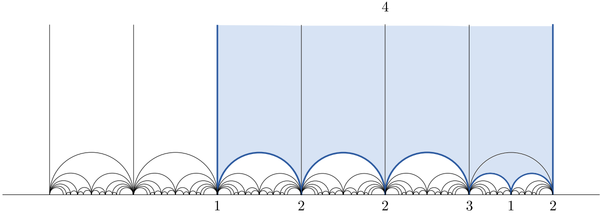


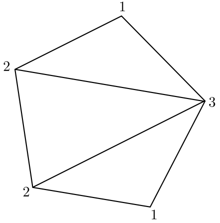




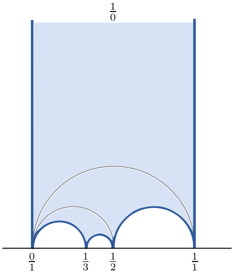
S. Morier-Genoud, V. Ovsienko & S. Tabachnikov, $\mathrm{SL}_2(\mathbb{Z})$-*tilings of the torus, Coxeter–Conway friezes and Farey triangulations*, 2015

Proving the Conway–Coxeter theorem

Theorem (Conway & Coxeter) There is a one-to-one correspondence between positive integer friezes and triangulated polygons.



0	1	1	1	1	0					
	0	1	2	3	1	0				
		0	1	2	1	1	0			
			0	1	1	2	1	0		
				0	1	3	2	1	0	
					0	1	1	1	1	0



Tame SL_2-tilings

Definition Recall that an SL_2-*tiling* is an infinite array of integers such that any two-by-two submatrix satisfies $ad - bc = 1$.

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 5 & 9 & 4 & 7 & 17 & \\ & 1 & 2 & 1 & 2 & 5 & \\ \cdots & 2 & 5 & 3 & 7 & 18 & \cdots \\ & 1 & 3 & 2 & 5 & 13 & \\ & 3 & 10 & 7 & 18 & 47 & \\ & & & \vdots & & & \end{array}
\qquad
\begin{array}{ccccccc} & & & \vdots & & & \\ & -13 & -8 & \color{red}{-3} & -4 & -5 & \\ & -8 & -5 & \color{red}{-2} & -3 & -4 & \\ \cdots & \color{red}{-3} & \color{red}{-2} & \color{red}{-1} & \color{red}{-2} & \color{red}{-3} & \cdots \\ & -4 & -3 & \color{red}{-2} & -5 & -8 & \\ & -5 & -4 & \color{red}{-3} & -8 & -13 & \\ & & & \vdots & & & \end{array}$$

Tame SL_2-tilings

Definition Recall that an SL_2-*tiling* is an infinite array of integers such that any two-by-two submatrix satisfies $ad - bc = 1$.

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 5 & 9 & 4 & 7 & 17 & \\ & 1 & 2 & 1 & 2 & 5 & \\ \cdots & 2 & 5 & 3 & 7 & 18 & \cdots \\ & 1 & 3 & 2 & 5 & 13 & \\ & 3 & 10 & 7 & 18 & 47 & \\ & & & \vdots & & & \end{array} \qquad \begin{array}{ccccccc} & & & \vdots & & & \\ & -13 & -8 & -3 & -4 & -5 & \\ & -8 & -5 & -2 & -3 & -4 & \\ \cdots & -3 & -2 & -1 & -2 & -3 & \cdots \\ & -4 & -3 & -2 & -5 & -8 & \\ & -5 & -4 & -3 & -8 & -13 & \\ & & & \vdots & & & \end{array}$$

Definition An SL_2-tiling is *tame* if the determinant of each three-by-three submatrix is 0.

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Theorem Positive integer SL_2-tilings are tame.

Tame SL_2-tilings

Theorem An SL_2-tiling is tame if and only if there are integers k_i, $i \in \mathbb{Z}$, such that

$$\mathsf{row}_{i+1} + \mathsf{row}_{i-1} = k_i \, \mathsf{row}_i, \quad \text{for } i \in \mathbb{Z}.$$

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Comments on tame tilings

- This row recurrence relation resembles the continued fractions recurrence relation.
- Tame tilings can have zeros and negative integers.
- Tame tilings have rigidity.
- More general tilings unknown.

Classification of tame SL_2-tilings

Theorem There is a one-to-one correspondence between tame SL_2-tilings and pairs of biinfinite paths in the Farey graph.

F. Bergeron & C. Reutenauer, *SL_k-tilings of the plane*, 2010

Classification of tame SL_2-tilings

Theorem There is a one-to-one correspondence between tame SL_2-tilings and pairs of biinfinite paths in the Farey graph.

Remark Really we consider tame SL_2-tilings modulo $\pm$, and we consider pairs of biinfinite paths in the Farey graph modulo the action of the modular group.

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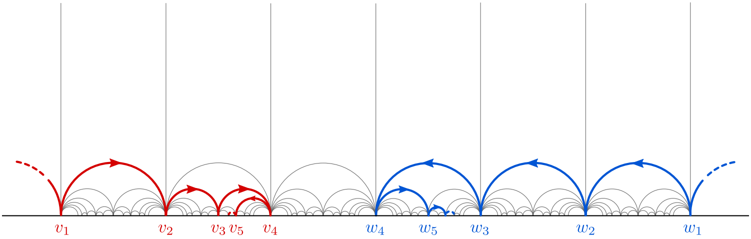
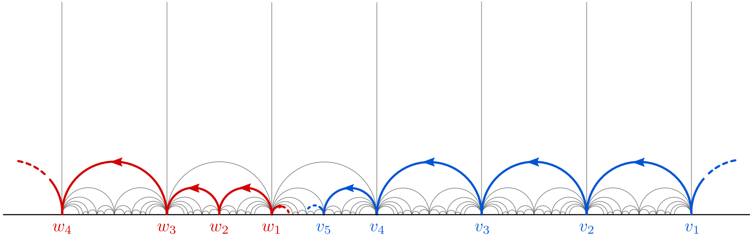


F. Bergeron & C. Reutenauer, *SL_k-tilings of the plane*, 2010

Classification of positive integer SL_2-tilings

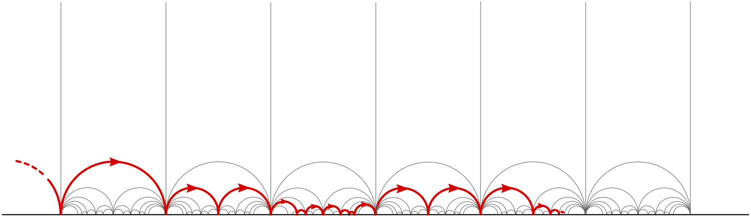
Theorem There is a one-to-one correspondence between positive integer SL_2-tilings and pairs of monotonic biinfinite paths in the Farey graph that do not intersect.



C. Bessenrodt, P. Jørgensen & T. Holm, *All SL_2-tilings come from infinite triangulations*, 2017

Classification of infinite friezes

Theorem There is a one-to-one correspondence between infinite friezes and biinfinite paths in the Farey graph.

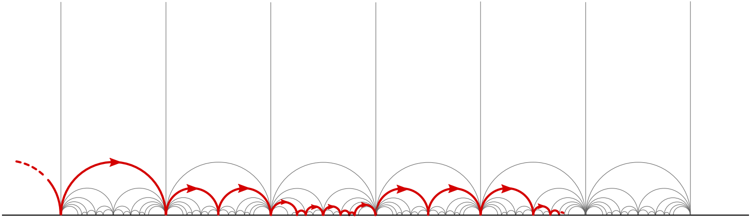


K. Baur, M.J. Parsons & M. Tschabold, *Infinite friezes*, 2016

Classification of infinite friezes

Theorem There is a one-to-one correspondence between infinite friezes and biinfinite paths in the Farey graph.

Corollary There is a one-to-one correspondence between positive integer infinite friezes and monotonic biinfinite paths in the Farey graph.

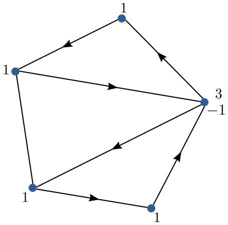


K. Baur, M.J. Parsons & M. Tschabold, *Infinite friezes*, 2016

Classification of tame friezes

Theorem There is a one-to-one correspondence between tame integer friezes of period n and closed paths in the Farey graph of length n.

	0		0		0		0		0		0		0		0		
		1		1		1		1		1		1		1		1	
	1		1		-1		1		1		3		1		1		
$\cdots$		2		0		-2		-2		0		2		2		0	$\cdots$
	-1		-1		-3		-1		-1		1		-1		-1		
		-1		-1		-1		-1		-1		-1		-1		-1	
	0		0		0		0		0		0		0		0		



The Farey graph modulo n

Farey graph modulo n

Definition Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n\}$.
The *Farey graph modulo* n is the graph with vertices

$$\{(a, b) : a, b \in \mathbb{Z}_n, \mathsf{gcd}(a, b, n) = 1\}/\sim,$$

where $(a, b) \sim (a', b')$ if $(a', b') \equiv -(a, b) \pmod{n}$, and such that vertices (a, b) and (c, d) are joined by an edge if and only if $ad - bc \equiv \pm 1 \pmod{n}$.

I. Ivrissimtzis & D. Singerman, *Regular maps and principal congruence subgroups of Hecke groups*, 2005

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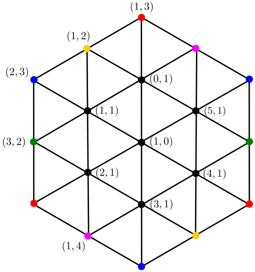

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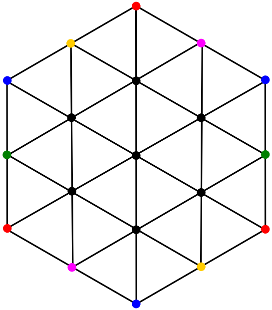
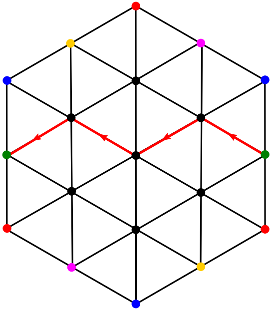
Ivrissimtzis and Singerman's regular maps

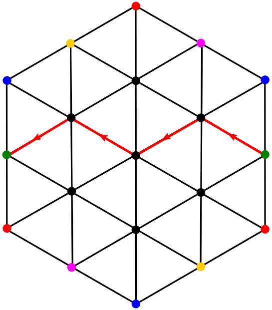


Ivrissimtzis and Singerman's regular maps



$$
\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & 2 & & 4 & & 2 & & 4 & & 2 & & \cdots \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Ivrissimtzis and Singerman's regular maps



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 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
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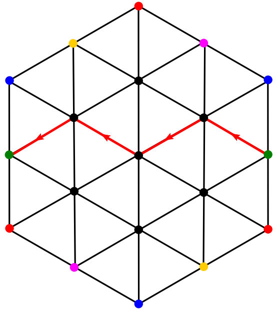
Definition The vertices of the form $(a, 0)$ are said to be a set of *poles*, as is any image of this set under $\Gamma(n)$. There are $\phi(n)/2$ vertices in a set of poles.



$$\begin{array}{ccccccccccccc}
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 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
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Theorem There is a one-to-one correspondence between tame integer friezes modulo n and paths on the Farey graph modulo n from one vertex in a set of poles to another.

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