DISCREPANCY ESTIMATES FOR GENERALIZED POLYNOMIALS

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Please Publish in ACADEMIC Journals!

like

NWEJM — North Western European Journal of Mathematics

Walter Craig, Sinnou David, Jean-Pierre Demailly, Florian Luca, Kaisa Matomäki, OR, Jie Wu,...

FUNCTIONES ET APPROXIMATIO — Little brother of Acta Arithmetica Euclid Project

Jerzy Kaczorowski, Julian Musielak, Leszek Skrzypczak,...

Several years of existence, Referenced in Mathscinet and Zentralblatt
HRI Lecture Notes Series, IMSc (Chennai) Lecture Notes in Mathematics

Initiation

In 2008 (Green & Tao, 2012):

The Möbius function is

strongly orthogonal to Generalized Polynomials ~~*nilsequences*~~

Generalized Polynomials = Bracket Polynomials

Polynomials like $\{n^3\sqrt{2} + n^7[n\sqrt{5} + n^2] + \sqrt{\pi}\}$

- $[x]$ = integer part (floor) of x.
- $\{x\}$ = fractional part of x.

Strange beasts: $[(n+1)\pi] - [n\pi] - 3$, $\frac{2}{3}\{\pi n\} + \frac{1}{3}[2\{\pi n\}]$

New information about $\mu(n)$, but what is it?

Project: Using Generalized Polynomials

- Chennai 2011: Working group with Anirban Mukhopadhyay

  We want precise and flexible informations

  With $\{\alpha n\}$, we have Fourier analysis and localisation. We seek similar geometrical understanding.

- Linz 2015: Roswitha Hofer interested in subsequences.

- 2016: G. Kasi Viswanadham joined in.

Equidistribution

Theorem (Folklore)

$n \mapsto \{\beta[\alpha n]\}$ is equidistributed modulo 1 iff $1, \alpha, \alpha\beta$ are linearly independent / $\mathbb{Q}$.

Theorem ((Veech, 1971), (Håland, 1993))

$$P(x) = x^d + a_1 x^{d-1} + \ldots + a_d,\ \alpha\beta \neq 0$$

- *When $a_1, \ldots, a_d$ rational, $\{\beta[\alpha P(n)]\}$ equidistributed iff $1, \alpha, \alpha\beta$ lin. indep./$\mathbb{Q}$.*
- *Else $\{\beta[\alpha P(n)]\}$ equidistributed.*

Diophantine Background

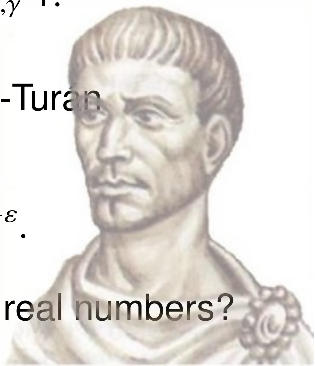
$(t+1)$ is an irrationality measure of $\gamma \in \mathbb{R}$ when

$$\min_{m \in \mathbb{Z} \setminus \{0\}} |m|^{t+\varepsilon} \| m\gamma \| \gg_{\varepsilon,\gamma} 1.$$

In which case, we have (via Erdős-Turán inequality):

$$D_N(n\alpha) \ll_{\alpha,\varepsilon} N^{-1/t+\varepsilon}.$$

Corresponding notion for a pair of real numbers?



We follow (Niederreiter, 1973):

$(\gamma,\delta)\in\mathbb{R}^2$ is of *finite type t* when

$$\min_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\Big((1+|m|)(1+|n|)\Big)^{t+\varepsilon}\|m\gamma+n\delta\|\gg_\varepsilon 1.$$

1. Let γ, δ real algebraic numbers, $1,\gamma,\delta$ are lin. indep./$\mathbb{Q}$. (Schmidt, 1967): (γ,δ) is of finite type 1.
2. For almost all $(\gamma,\delta)\in\mathbb{R}^2$, the pair (γ,δ) is of finite type 1.

To accomodate usage, Niederreiter proposed to us:

(γ, δ) is of *weak cotype* $t' \geqslant 1$ when, for any $\varepsilon > 0$,

$$\min_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (|m| + |n|)^{t'+\varepsilon} \| m\gamma + n\delta \| \gg_\varepsilon 1.$$

weak cotype $t' \implies$ finite type t' $\quad |m| + |n| \leqslant (1 + |m|)(1 + |n|)$

(Hata, 1992):

$$\min_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (|m| + |n|)^{7.0161} \| m\pi + n\log 2 \| \gg 1.$$

Other examples??

Discrepancy, I

Theorem (R. Hofer & OR (2015))

Let α, β real numbers

- $1, \alpha, \alpha\beta$ *lin. indep./$\mathbb{Q}$ (automatic).*
- $(\alpha, \alpha\beta)$ *and* $(\beta, 1/\alpha)$ *of finite type t.*

Then, for every $\varepsilon > 0$,

$$D_N([n\alpha]\beta) \ll_{\alpha,\beta,\varepsilon} N^{-1/(3t-2)+\varepsilon}.$$

That's $1/N^{1-\varepsilon}$ when $t = 1$! *Optimal??*

With weak cotype, a hypothesis either on $(\alpha, \alpha\beta)$ or on $(\beta, 1/\alpha)$ is enough.

Corollary

For almost all pairs of real numbers (α, β) in the sense of Lebesgue measure we have for every $\varepsilon > 0$ that $D_N([n\alpha]\beta) \ll_{\alpha,\beta,\varepsilon} N^{-1+\varepsilon}$.

Corollary

Discrepancy of $([n/\pi]\log 2)$ and $([n/\log 2]\pi)$ is $\ll N^{-0.052498}$.

Omega Estimates??

Equidistribution, II

(Bergelson & Leibman, 2007)

- Dynamical understanding of Generalized Polynomials.
- Description of which measure makes a Generalized Polynomials well-distributed.
- The strange beasts may be more telling . . .

Ohio State University!

Discrepancy, II

Back to India: let us look at $\{\beta[\alpha n^2]\}$ which I thought would be very difficult.

Theorem (A. Mukhopadhyay & OR & G. Kasi Viswanadham – 2018)

Let $(\alpha, \alpha\beta)$ *of finite type* $t \geqslant 1$. *For* $\epsilon > 0$,

$$D_N([P(n)\alpha]\beta) \ll_{\epsilon,\alpha,\beta,d} N^{-\frac{2-2^{-d+2}}{2^{d-1}(2t+1)+7t+2}+\epsilon}.$$

$P(n)$ monic degree $d \geqslant 2$.

$d = 2 \rightarrow \ll N^{-\frac{1}{11t+4}}$, $d = 3 \rightarrow \ll N^{-\frac{1}{10t+4}}$, $d = 4 \rightarrow \ll N^{-\frac{7}{92t+40}}$.

We do not have metric results, though that may change. Or Omega estimate

Lemma ((Vinogradov, 1927))

Let α of irrationality meas. $t+1$. For any $\epsilon > 0$.

$$D_N(P(n)\alpha) \ll_{\epsilon,d,t} N^{-\frac{2d}{(1+t)2^d}+\epsilon} + N^{-\frac{1}{2^{d-1}}+\epsilon}$$

$P(n)$ monic degree $d \geqslant 2$.

Ancestors? $d = 2$ in (Behnke, 1922)
Successors???

With (Vinogradov, 2004), the exponent becomes $-E$ where

$$E \gg \frac{1}{(d^2 \log d)(t + d^2 \log d)}.$$

Elements of proof

For $\tau \in \mathbb{R}$, let $f_\tau(x) = e(\tau\{x\})$.
For $\delta > 0$,

$$g_{\tau,\delta}(x) = \frac{1}{(2\delta)^r} 1_{[-\delta,\delta]} * \cdots * 1_{[-\delta,\delta]} * f_\tau(x),$$

Lemma ($g_{\tau,\delta}$ approximates f_τ)

For any $\{u_n\}_{n\geqslant 1}$, and any N we have

$$\sum_{n\leqslant N} |f_\tau(u_n) - g_{\tau,\delta}(u_n)| \ll Nr\delta + Nr^2\delta|\tau| + ND_N(u_n).$$

First key

$$\hat{g}_{\tau,\delta}(k) = \left(\frac{\sin 2\pi k\delta}{2\pi k\delta}\right)^r \frac{e(\tau + k) - 1}{2\pi i(\tau + k)}$$

!

Lemma

Let $\tau, \delta > 0$, $p > 1$, *then* $\sum_{k\in\mathbb{Z}} |\hat{g}_{\tau,\delta}(k)|^p \ll_p 1.$

Uniform in $|\tau| \leqslant 1/(2\delta)$

+ more usual lemmas on exponential sums with polynomial phase.

Skeleton

- $\beta[\alpha P(n)] = \beta\alpha P(n) - \beta\{\alpha P(n)\}$.
- Erdős-Turán inequality.
- $ND_N([P(n)\alpha]\beta) \ll$
$$\sum_{h=1}^{H}\frac{1}{h}\left|\sum_{k\in\mathbb{Z}}\hat{g}_{-h\beta,\delta(h)}(k)\sum_{n=0}^{N-1}e\Big(P(n)\alpha(h\beta-k)\Big)\right| + \varepsilon$$

  EXPLICIT dep. in h and k !
- *We use only process A. But we have exponential sums with parameters!*

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