Uniform distribution for zeros of random polynomials

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Real zeros of random polynomials

Let $\{A_k\}_{k=0}^{\infty}$ be random variables on the space $(\mathbb{C}, \mathcal{B}, \nu)$, with $\mathbb{P}(A_k \in S) = \nu(S)$ and $\mathbb{E}[A_k] = \int x \, d\nu(x)$. For $P_n(x) = \sum_{k=0}^{n} A_k x^k$, let $N_n(E)$ be the number of zeros of P_n in a set $E \subset \mathbb{C}$.

Bloch&Pólya, 1932: $\mathbb{E}[N_n(\mathbb{R})] = O(\sqrt{n})$ for $A_k \in \{-1, 0, 1\}$
Littlewood&Offord, 1930-40's: $\mathbb{P}(N_n(\mathbb{R}) \le C \log^2 n) = 1 - o(1)$ as $n \to \infty$. Similar results were obtained by Erdős&Offord and others.

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Assumptions on coefficients: Erdős&Offord, 1956; Stevens, 1969; Logan&Shepp, 1968; Ibragimov&Maslova, 1971; Do&Nguyen&Vu, 2015
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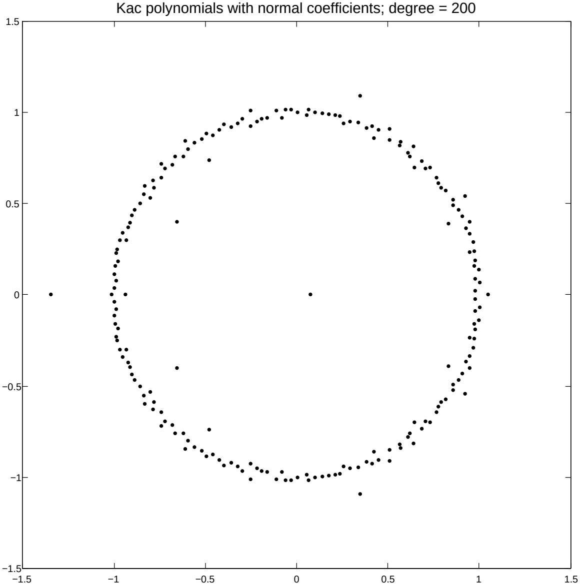
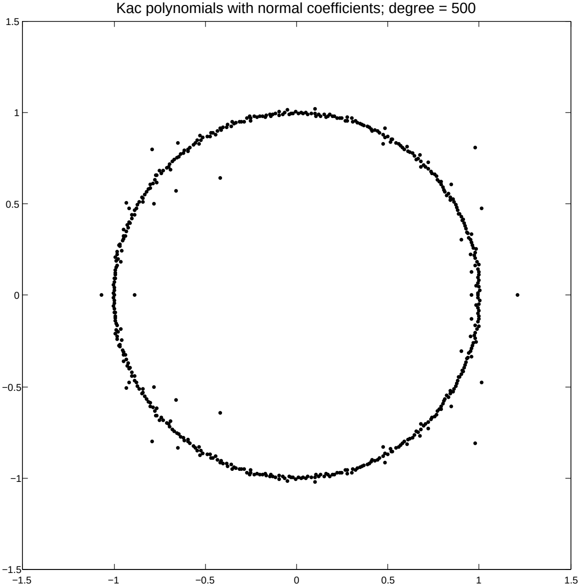
Gaussian coefficients, degree 200



Gaussian coefficients, degree 500



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Question: When $\tau_n \stackrel{w}{\to} \mu_{\mathbb{T}}$ with probability one (a.s.)?

Selected results: Hammersley, 1956; Shparo&Shur, 1962; Arnold, 1966; Shepp&Vanderbei, 1995; Ibragimov&Zeitouni, 1997; Shiffman&Zelditch, 2003; Bloom, 2007; Hughes&Nikeghbali, 2008

C1 $A_k \in \mathbb{C}$ are i.i.d. r.v. with $\mathbb{P}(A_0 = 0) < 1$ and $\mathbb{E}[\log^+ |A_0|] < \infty$.

Ibragimov and Zaporozhets, 2013: $\tau_n \stackrel{w}{\to} \mu_{\mathbb{T}}$ a.s. $\Leftrightarrow$ **C1**

Arnold, 1966: **C1** $\Leftrightarrow \limsup_{n\to\infty} |A_n|^{1/n} = 1$ a.s.
Hence the radius of convergence for $\sum_{n=0}^\infty A_n z^n$ is 1 a.s.

C2 $A_k \in \mathbb{C}$ are identically distributed r.v. with $\mathbb{E}[|\log |A_0||] < \infty$.

Pritsker, 2014: **C2** $\Rightarrow \tau_n \stackrel{w}{\to} \mu_{\mathbb{T}}$ a.s.

Remark: **C2** $\Leftrightarrow \lim_{n\to\infty} |A_n|^{1/n} = 1$ a.s.

Uniform zero distribution of random polynomials

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Expected discrepancy

Use Erdős&Turán (1950) and generalizations. Let
$A_r(\alpha,\beta)=\{z : r<|z|<1/r,\ \alpha\le\arg z<\beta\},\ 0<r<1.$

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$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta))-\frac{\beta-\alpha}{2\pi}\right|\right]\le C_r\sqrt{\frac{1}{tn}\log\sum_{k=0}^n\mathbb{E}[|A_k|^t]-\frac{1}{2n}\mathbb{E}[\log|A_0A_n|]}.$$

If $\max_{0\le k\le n}\mathbb{E}[|A_{k,n}|^t]<C$ and $\min_{k=0\&n}\mathbb{E}[\log|A_{k,n}|]>c\ \forall\, n\in\mathbb{N}$, then

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$$\mathbb{E}[N_n(A_r(\alpha,\beta))]=\frac{\beta-\alpha}{2\pi}n+O\left(\sqrt{n\log n}\right).$$

If $E\subset\mathbb{C}$ satisfies $\mathrm{dist}(E,\mathbb{T})>0$, then $\mathbb{E}[N_n(E)]=O(\log n)$.
If E is a polygon inscribed in $\mathbb{T}$, then $\mathbb{E}[N_n(E)]=O\left(\sqrt{n\log n}\right)$.

Expected discrepancy

Use Erdős&Turán (1950) and generalizations. Let
$A_r(\alpha,\beta) = \{z : r < |z| < 1/r,\ \alpha \le \arg z < \beta\},\ 0 < r < 1.$

Pritsker, 2014: If $\{A_k\}_{k=0}^n$ satisfy. $\mathbb{E}[|A_k|^t] < \infty,\ k = 0,\dots,n$, for a fixed $t > 0$, and $\mathbb{E}[\log|A_0|] > -\infty$, $\mathbb{E}[\log|A_n|] > -\infty$, then

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|\right] \le C_r \sqrt{\frac{1}{tn}\log\sum_{k=0}^n \mathbb{E}[|A_k|^t] - \frac{1}{2n}\mathbb{E}[\log|A_0A_n|]}.$$

If $\max_{0\le k\le n}\mathbb{E}[|A_{k,n}|^t] < C$ and $\min_{k=0\&n}\mathbb{E}[\log|A_{k,n}|] > c\ \forall\, n\in\mathbb{N}$, then

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Zeros of lacunary random polynomials

Consider lacunary polynomials $L_n(z) = \sum_{k=0}^{n} A_k z^{r_k}$, where $\{r_k\}_{k=0}^{\infty} \subset \mathbb{N}$ are increasing and $\{A_n\}_{n=0}^{\infty} \subset \mathbb{C}$ are random variables.

Pritsker, 2018: Let $a > 0$ and $p \geq 1$. Suppose either $\{A_n\}_{n=0}^{\infty}$ are non-trivial i.i.d. random variables satisfying $\mathbb{E}[(\log^+ |A_n|)^{1/p}] < \infty$, or $\{A_n\}_{n=0}^{\infty}$ are identically distributed and $\mathbb{E}[|\log |A_n||^{1/p}] < \infty$. If $r_n \sim an^p$ then $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

Assume that $\liminf_{n\to\infty} r_n^{1/n} > 1$. If $\{A_n\}_{n=0}^{\infty} \subset \mathbb{C}$ are identically distributed and $\mathbb{E}[\log^+ |\log |A_n||] < \infty$, then $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

Let $\{A_n\}_{n=0}^{\infty}$ be identically distributed with $\mathbb{E}[|A_n|^t] < \infty$ for a fixed $t \in (0, 1]$, and $\mathbb{E}[\log |A_n|] > -\infty$. If $\liminf_{n\to\infty} r_n^{1/n} = q > 1$ then

$$\limsup_{n\to\infty} \left| \tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi} \right|^{1/n} \leq \frac{1}{\sqrt{q}} \quad \text{a.s.}$$

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Random polynomials spanned by general bases

Let $E \subset \mathbb{C}$ be compact, $\text{cap}(E) > 0$, with the equilibrium measure μ_E. Define $\|P_n\|_E := \sup_{z \in E} |P_n(z)|$. Let $B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j$, where $b_{j,k} \in \mathbb{C}$ and $b_{k,k} \neq 0$ for $k = 0, 1, 2, \ldots$. We assume that

$$\limsup_{k\to\infty} ||B_k||_E^{1/k} \leq 1 \quad \text{and} \quad \lim_{k\to\infty} |b_{k,k}|^{1/k} = 1/\text{cap}(E).$$

Applies to many orthonormal polynomials and to other bases.

Pritsker, 2014-15: Suppose that E has empty interior and connected complement. If $\{A_k\}_{k=0}^{\infty}$ satisfy either **C1** or **C2**, then the zero counting measures for $P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$ satisfy $\tau_n \xrightarrow{w} \mu_E$.

Example: $E = [a, b] \subset \mathbb{R}$ and $\{B_k\}_{k=0}^{\infty}$ are orthonormal w.r.t. a measure ν supported on E such that $\nu' > 0$ a.e. on $[a, b]$, with

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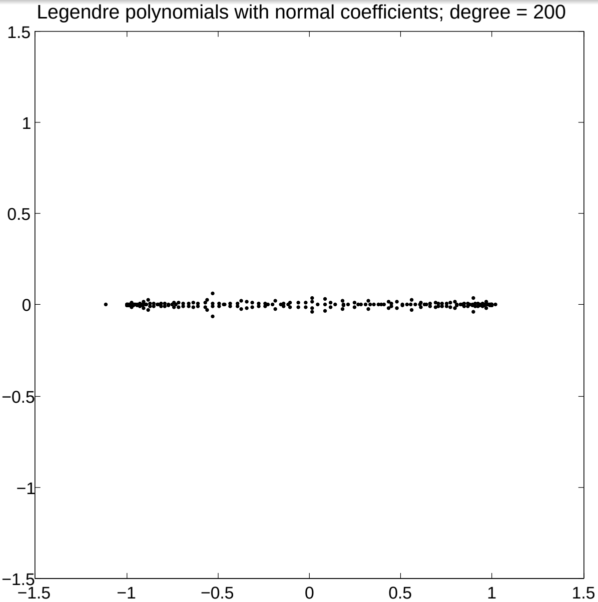
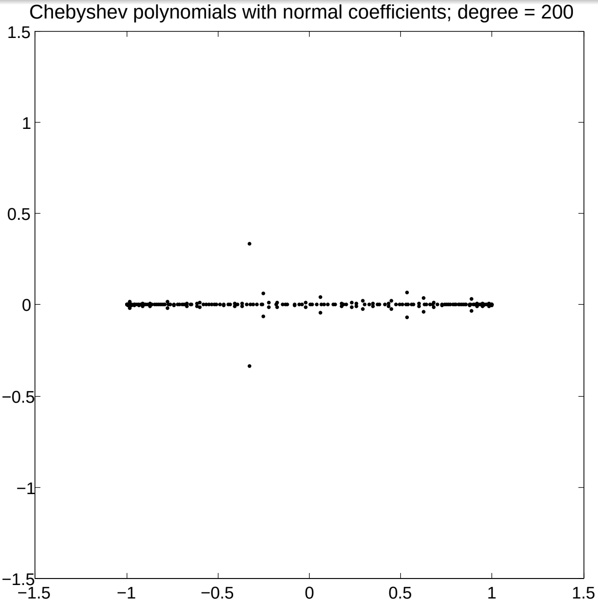
Random Legendre, $\mathcal{N}(0,1)$ coefficients, deg=200



Random Chebyshev, $\mathcal{N}(0,1)$ coefficients, deg=200



Real zeros of random orthogonal polynomials on $\mathbb{R}$

Let $\{A_k\}_{k=0}^{\infty}$ be i.i.d. real Gaussian random variables and $\{p_k\}_{k=0}^{\infty}$ be orthonormal Legendre polynomials. Consider $P_n(x) = \sum_{k=0}^{n} A_k p_k(x)$.

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Questions: Error terms in asymptotic formulas, random coefficients with more general distributions, almost sure convergence.

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and for $[a,b] \subset E \setminus S$ we have

$$\lim_{n\to\infty} \mathbb{E}\left[N_n\left([a,b]\right)\right]/n = \mu_E([a,b])/\sqrt{3}.$$

Questions: Error terms in asymptotic formulas, random coefficients with more general distributions, almost sure convergence.