Tractability properties of the weighted star discrepancy

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Joint work with
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Weighted star discrepancy

For $\mathcal{P} = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{N-1}\}$ in $[0,1)^d$ the **local discrepancy** is

$$\Delta_{\mathcal{P}}(\boldsymbol{\alpha}) := \frac{\mathcal{P} \cap [\mathbf{0}, \boldsymbol{\alpha})}{N} - \mathrm{Volume}([\mathbf{0}, \boldsymbol{\alpha})).$$

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Star discrepancy of $\mathcal{P}$:

$$D_N^*(\mathcal{P}) = \sup_{\boldsymbol{\alpha} \in [0,1]^d} |\Delta_{\mathcal{P}}(\boldsymbol{\alpha})|.$$

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$$(\boldsymbol{\alpha}_{\mathfrak{u}}, \mathbf{1}) = (y_1, \ldots, y_d) \quad \text{where} \quad y_j = \begin{cases} \alpha_j & \text{if } j \in \mathfrak{u}, \\ 1 & \text{if } j \notin \mathfrak{u}. \end{cases}$$

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$\boldsymbol{\gamma}$-weighted star discrepancy of $\mathcal{P}$:

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) := \sup_{\boldsymbol{\alpha} \in [0,1]^d} \max_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{\mathfrak{u}} |\Delta_{\mathcal{P}}(\boldsymbol{\alpha}_{\mathfrak{u}}, \mathbf{1})|$$

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$\boldsymbol{\gamma}$-weighted star discrepancy of $\mathcal{P}$:

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) := \sup_{\boldsymbol{\alpha} \in [0,1]^d} \max_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{\mathfrak{u}} |\Delta_{\mathcal{P}}(\boldsymbol{\alpha}_{\mathfrak{u}}, \mathbf{1})|$$

If $\gamma_j = 1$ for all $j \geq 1$, then $D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) = D^*_N(\mathcal{P})$.

Weighted star discrepancy and multivariate integration

Let

$$\mathcal{W}_1^{\mathbf{1}} := \mathcal{W}_1^{(1,1,\ldots,1)}([0,1]^d)$$

be the Sobolev space of functions defined on $[0,1]^d$ that are

- once differentiable in each variable, and
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- whose derivatives have finite L_1 norm.

Consider

$$\mathcal{F}_{d,1,\gamma} = \{f \in \mathcal{W}_1^{\mathbf{1}} \ : \ \|f\|_{d,1,\gamma} < \infty\},$$

where

$$\|f\|_{d,1,\gamma} = |f(\mathbf{1})| + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \frac{1}{\gamma_{\mathfrak{u}}} \left\| \frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}}, \mathbf{1}) \right\|_{L_1}.$$

Weighted star discrepancy and multivariate integration

QMC integration in $\mathcal{F}_{d,1,\gamma}$: the **worst-case error** of a QMC rule is

$$e(\mathcal{P};\mathcal{F}_{d,1,\gamma}) = \sup_{\substack{f\in\mathcal{F}_{d,1,\gamma}\\ \|f\|_{d,1,\gamma}\le 1}} \left| \int_{[0,1]^d} f(\mathbf{t})\,\mathrm{d}\mathbf{t} - \underbrace{\frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)}_{\text{QMC rule}} \right|.$$

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- small $\gamma_{\mathfrak{u}}$ forces $\|\frac{\partial^{|\mathfrak{u}|}}{\partial \mathbf{x}_{\mathfrak{u}}} f(\mathbf{x}_{\mathfrak{u}},\mathbf{1})\|_{L_1}$ to be small

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Theorem (Sloan & Woźniakowski; Koksma-Hlawka)

$$e(\mathcal{P};\mathcal{F}_{d,1,\gamma}) = D^*_{N,\gamma}(\mathcal{P})$$

Tractability

For $d, N \in \mathbb{N}$ the N**th minimal weighted star discrepancy** is

$$\mathrm{disc}_{\boldsymbol{\gamma}}(N,d) = \inf_{\substack{\mathcal{P} \subseteq [0,1)^d \\ \#\mathcal{P}=N}} D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}).$$

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For $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$ the **inverse of the weighted star discrepancy** is

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We are interested in the behavior of $N^*_{\boldsymbol{\gamma}}(\varepsilon,d)$ for $\varepsilon \to 0$ and $d \to \infty$.

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Notions of **tractability** help to classify the dependence of $N^*_{\gamma}(\varepsilon, d)$ on

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1. **Polynomial tractability (PT)**: if $\exists\ C, \alpha, \beta > 0$:

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The infimum of $\alpha > 0$ for which (1) holds is the ε-**exponent** of SPT.

Existing results for the unweighted star discrepancy

For $\gamma_{\mathfrak{u}} = 1$ for all $\mathfrak{u} \subseteq [d]$:

Theorem (Heinrich, Novak, Wasilkowski, Woźniakowski, 2001)

$$\mathrm{disc}_{\mathbf{1}}(N, d) \leq C\sqrt{\frac{d}{N}} \quad \text{for all} \quad N, d \in \mathbb{N}$$

and hence

$$N_{\mathbf{1}}^{*}(\varepsilon, d) \leq C^2 d \varepsilon^{-2}$$

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- the exact dependence of $N_{\mathbf{1}}^{*}(\varepsilon, d)$ on $\begin{cases} d & : \text{ linear} \\ \varepsilon^{-1} & : \text{ still open} \end{cases}$
- **Aistleitner:** $C = 10$ and hence $C^2 = 100$
- **Gnewuch & Hebbinghaus:** $C = 2.5287$ and hence $C^2 = 6.3943\ldots$

Existing results for the weighted star discrepancy

Theorem (Hickernell, Sloan, Wasilkowski, 2004)

If $\exists \tau > 0$ such that $\sum_j \gamma_j^\tau < \infty$, then

$$\mathrm{disc}_{\boldsymbol{\gamma}}(N, d) \leq C(\delta, \tau) \frac{1}{N^{1/2-\delta}} \quad \text{for all} \quad d, N \in \mathbb{N}.$$

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- H.S.W. conjectured that this condition might be necessary for SPT

Existing results for the weighted star discrepancy

Theorem (Aistleitner, 2014)

If $\exists c > 0$ such that $\sum_j \exp(-c\gamma_j^{-2}) < \infty$, then

$$\mathrm{disc}_{\boldsymbol{\gamma}}(N, d) \ll_{\boldsymbol{\gamma}} \frac{1}{\sqrt{N}} \quad \text{for all} \quad d, N \in \mathbb{N}.$$

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- e.g.

$$\gamma_j = \frac{1}{\sqrt{\log j}}$$

Existing results for the weighted star discrepancy

Theorem (Dick, Leobacher, Pil., 2005)

If $\sum_j \gamma_j < \infty$, then one can construct (CBC) a polynomial lattice point set $\mathcal{P}$ over $\mathbb{F}_p$ such that

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- $\mathcal{P} = \mathcal{P}(\boldsymbol{\gamma})$ depends on the weights
- fast CBC construction with $O(dN \log N)$ operations

Existing results for the weighted star discrepancy

Theorem (Wang, 2002)

Let $\mathcal{P}$ be the first N elements of an d-dimensional Niederreiter sequence in prime-power base q. Then

$$D^*_{N,\gamma}(\mathcal{P}) \le \frac{1}{N} \max_{\emptyset \neq \mathfrak{u} \subseteq [d]} \prod_{j \in \mathfrak{u}} \left[\gamma_j (C\, j \log(j+q) \log(qN))\right],$$

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- similar results for Sobol' and Halton sequences
- further results for so-called p-sets and sequences that are based on certain RNG (Dick, Gomez-Perez, Pil., Winterhof)

Halton sequence

Let $b \geq 2$. For $n = n_0 + n_1 b + n_2 b^2 + \cdots$ define

$$\varphi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \cdots .$$

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Halton sequence

Let $b_1, b_2, b_3, \ldots$ be the prime numbers in increasing order. Then

$$\mathcal{H}_{b_1,\ldots,b_d} = (\mathbf{x}_n)_{n \geq 0} \quad \text{where} \quad \mathbf{x}_n = (\varphi_{b_1}(n), \ldots, \varphi_{b_d}(n)).$$

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Wang 2002

$$\sum_j \gamma_j (j \log j) < \infty \Rightarrow \text{SPT with } \varepsilon\text{-exponent } 1$$

Halton sequence

Theorem (Hinrichs, Pill., Tezuka 2018)

- If
$$\sum_{j\geq 1} j\gamma_j < \infty, \qquad \text{e.g. } \gamma_j = \frac{1}{j^{2+\delta}},$$
then the weighted star discrepancy of the Halton sequence $\mathcal{H}_{b_1,\ldots,b_d}$ achieves SPT with ε-exponent 1.

Halton sequence

Theorem (Hinrichs, Pill., Tezuka 2018)

- If
$$\sum_{j\ge 1} j\gamma_j < \infty, \qquad \text{e.g. } \gamma_j = \frac{1}{j^{2+\delta}},$$
then the weighted star discrepancy of the Halton sequence $\mathcal{H}_{b_1,\ldots,b_d}$ achieves SPT with ε-exponent 1.

- If
$$\sup_{d\ge 1} \max_{\emptyset\ne\mathfrak{u}\subseteq[d]} \prod_{j\in\mathfrak{u}} (j\gamma_j) < \infty, \qquad \text{e.g. } \gamma_j = \frac{1}{j},$$
then the weighted star discrepancy of the Halton sequence $\mathcal{H}_{b_1,\ldots,b_d}$ achieves SPT with ε-exponent at most 2.

Halton sequence

Proof: Unweighted star discrepancy:

$$D_N^*(\mathcal{H}_{b_1,\ldots,b_d}(\mathfrak{u})) \leq \frac{(6\log N)^{|\mathfrak{u}|}}{N}\prod_{j\in\mathfrak{u}} j$$

Halton sequence

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Inserting in formula for $D_{N,\gamma}^*$:

- Case 1:

$$\begin{aligned} D_{N,\gamma}^*(\mathcal{H}_{b_1,\ldots,b_d}) &\le \frac{1}{N}\max_{\emptyset\neq\mathfrak{u}\subseteq[d]}\prod_{j\in\mathfrak{u}}(6j\gamma_j\log N) \\ &\le \frac{1}{N}\prod_{j=1}^{d}(1+6j\gamma_j\log N) \le \frac{c_\delta}{N^{1-\delta}}. \end{aligned}$$

  if $\sum_{j\ge1} j\gamma_j < \infty$.

Halton sequence

- Case 2: Trivially, $D_N^*(\mathcal{H}_{b_1,\ldots,b_d}(\mathfrak{u})) \le 1$.

$$D_{N,\gamma}^*(\mathcal{H}_{b_1,\ldots,b_d}) \quad \le \quad \max_{\emptyset \ne \mathfrak{u} \subseteq [d]} \prod_{j \in \mathfrak{u}} \gamma_j \min\left\{1, \frac{(6 \log N)^{|\mathfrak{u}|}}{N} \prod_{j \in \mathfrak{u}} j\right\}.$$

Halton sequence

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Then

$$\min\left\{1, \frac{(6 \log N)^{|\mathfrak{u}|}}{N} \prod_{j \in \mathfrak{u}} j \right\} \leq N^{-1+\frac{\log \log N^6}{W\left(\frac{6}{e}(\log N)^2\right)}} \prod_{j \in \mathfrak{u}} j,$$

where $W(x) \approx \log x - \log \log x + \frac{\log \log x}{\log x}$.

Summary

Existence results:

	weight condition	ε-exp.
H.N.W.W.	unweighted $\Rightarrow$ PT	≤ 2
H.S.W.	$\sum_j \gamma_j^\tau < \infty \Rightarrow$ SPT	≤ 2
A.	$\sum_j \exp(-c\gamma_j^{-2}) < \infty \Rightarrow$ SPT	≤ 2

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Constructive results:

	point set $\mathcal{P}$	weight condition	ε-exp.
D.L.P.	PLPS (CBC)	$\sum_j \gamma_j < \infty \Rightarrow$ SPT	1
D.P.	p-sets	$\sum_j \gamma_j < \infty \Rightarrow$ SPT	≤ 2
H.P.T.	Halton	$\sum_j j\gamma_j < \infty \Rightarrow$ SPT	1
		$\sup_d \max_{\mathfrak{u}} \prod_j j\gamma_j < \infty \Rightarrow$ SPT	≤ 2

Discussion

Consider $\gamma_j = 1/j^{1+\alpha}$ with $\alpha > 1$. Then $\sum_{j \geq 1} j\gamma_j < \infty$ and hence

$$D^*_{N,\gamma}(\mathcal{H}_{b_1,\ldots,b_d}) \leq \frac{c_\delta}{N^{1-\delta}}.$$

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$$D^*_{N,\gamma}(\mathcal{H}_{b_1,\ldots,b_d}) \leq \frac{c_\delta}{N^{1-\delta}}.$$

Closer look into the proof:

$$c_\delta \geq \left(1 + \frac{(\alpha-1)w^{\alpha-1}}{6}\right)^w$$

where

$$w \geq -1 + \left(\frac{6}{(\alpha-1)\delta}\left(1 + \frac{6}{\alpha-1}\right)\right)^{\frac{1}{\alpha-1}}$$

Discussion

δ	0.9	0.5	0.1	
$D^*_{N,\gamma} \lesssim$	$N^{-0.1}$	$N^{-0.5}$	$N^{-0.9}$	
c_δ	4×10^{35714}	10^{139333}	$10^{5152589}$	$\alpha = 1.5$
c_δ	5×10^{42}	$1,6 \times 10^{97}$	$1,7 \times 10^{775}$	$\alpha = 2$
c_δ	24.5	1129.5	1.7×10^{15}	$\alpha = 3$
c_δ	1.29	2.5	1922	$\alpha = 4$

Message

It is dangerous to write

$$D^*_{N,\gamma} \ll \frac{d^\beta}{N^\alpha} \qquad \text{or} \qquad N^*_\gamma(\varepsilon, d) \ll d^{\beta/\alpha} \varepsilon^{-1/\alpha}.$$

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Better:

$$D^*_{N,\gamma} \le C\frac{d^\beta}{N^\alpha} \quad \text{or} \quad N^*_\gamma(\varepsilon, d) \le C^{1/\alpha} d^{\beta/\alpha}\varepsilon^{-1/\alpha}$$

and study also the size of the constant C.

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and study also the size of the constant C.

Open problem

Improve the discrepancy bounds with respect to the implied constants.