# The Rudin-Shapiro sequence and similar sequences are normal along squares

#### Clemens Müllner



#### Tuesday, October 2nd, 2018

# Sum of digits

#### Let $q \ge 2$ be a base.

$$n=\sum_{j=0}^r \varepsilon_j^{(q)}(n)q^j,$$

where 
$$\varepsilon_j^{(q)}(n) \in \{0, \ldots, q-1\}$$
 and  $r = \lfloor \log_q(n) \rfloor$ .

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We define the sum of digits function  $s_q(n)$ 

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#### Let $q \ge 2$ and m such that gcd(q-1, m) = 1.

Theorem (Gelfond 1967)

 $(s_q(an + b) \mod m)_{n \in \mathbb{N}}$  is simply normal.

#### 2. Gelfond Problem

 $(s_q(p) \mod m)_{p \in \mathcal{P}}$  is simply normal. Solved by Mauduit and Rivat in 2010.

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#### Definition (Pattern-counting function)

Fix  $\ell \geq 1$  and a pattern  $P = (p_0, \dots, p_{\ell-1}) \in \{0, \dots, q-1\}^{\ell}, P \neq (0, \dots, 0).$ Then we define the *pattern-counting function* 

$$f_P(n) = \sum_{j=0}^{r-\ell+1} \mathbf{1}_{\left[\left(\varepsilon_{j+\ell-1}^{(q)}(n), \dots, \varepsilon_j^{(q)}(n)\right)=P\right]}.$$

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#### Definition (Block-additive function)

We say that  $b : \mathbb{N} \to \mathbb{Z}$  is *block-additive* / *digital* if there exists  $\ell \ge 1$  and  $F : \{0, \ldots, q-1\}^{\ell} \to \mathbb{Z}$  such that  $F(0, \ldots, 0) = 0$  and

$$b(n) = \sum_{j \in \mathbb{Z}} F(\varepsilon_{j+\ell-1}^{(q)}(n), \dots, \varepsilon_j^{(q)}(n)),$$

where 
$$\varepsilon_j(n) = 0$$
 for  $j \notin \{0, \ldots, r\}$ .

A block-additive function is (almost) a linear combination of pattern-counting functions.

#### Proposition

Block-additive functions mod m give automatic sequences.

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Digital sequences along squares are normal

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The Rudin-Shapiro sequence is simply normal along primes.

#### Theorem (Hanna, 2017)

Pattern-counting functions mod m along primes are simply normal.

#### Theorem (M., 2017)

All automatic sequences are orthogonal to the Mobius function. The densities along primes can be described.

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Let b be a block-additive function and  $m \in \mathbb{N}$  with gcd(q-1,m) = 1 and  $gcd(m, \{b(n), n \in \mathbb{N}\}) = 1$ . Then  $(b(n^2) \mod m)_{n \in \mathbb{N}}$  is normal in base m'.

This covers all pattern-counting functions mod m, where gcd(q-1,m) = 1, including the Thue-Morse sequence and the Rudin-Shapiro sequence.

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gcd(m, {b(n), n ∈ N}) = 1: necessary for simple normality.
gcd(q − 1, m) = 1: s<sub>q</sub>(n) ≡ n(mod q − 1) is periodic.
F(0,...,0) = 0 : f<sub>(0,...,0)</sub>(n) = ⌊log<sub>q</sub>(n)⌋ − ∑<sub>P≠(0,...,0)</sub> f<sub>P</sub>(n).
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- $gcd(m, \{b(n), n \in \mathbb{N}\}) = 1$ : necessary for simple normality.
- **2** gcd(q-1,m) = 1:  $s_q(n) \equiv n(mod q 1)$  is periodic.
- $F(0,...,0) = 0 : f_{(0,...,0)}(n) = \lfloor \log_q(n) \rfloor \sum_{P \neq (0,...,0)} f_P(n).$
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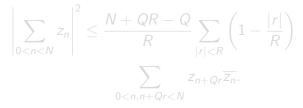
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# General Strategy

Rewrite the statement in terms of exponential sums.
Every block of length k appears with density m<sup>-k</sup> if

$$\left|\sum_{n\leq N} e\left(\sum_{j=0}^{k-1} \alpha_j b((n+j)^2)\right)\right| = o(N),$$

for all  $(\alpha_0, \ldots, \alpha_{k-1}) \neq (0, \ldots, 0)$ , where  $e(x) = exp(2\pi ix)$ . • Use a variation of the Van-der-Corput inequality,



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$$\left|\sum_{0 < n < N} z_n\right|^2 \le \frac{N + QR - Q}{R} \sum_{|r| < R} \left(1 - \frac{|r|}{R}\right)$$
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- Find appropriate estimates for the new Fourier-terms,

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