Distribution of short subsequences of inversive generator

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Inversive generator

Let $\mathcal{R}$ be a (finite, commutative, unitary) ring and define the sequence (u_n) by

$$u_n = \psi(u_{n-1}), \quad n \geq 1 \quad \text{with} \quad \psi(x) = \frac{a}{x} + b, \quad a, b \in \mathcal{R}.$$

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For computational aspect, there are two interesting cases:

- $\mathcal{R} = \mathbb{F}_q$, typically prime field: q is large prime.
- $\mathcal{R} = \mathbb{Z}_{p^t}$, p is small, typically $p = 2$.

Examples for inversive generator

Let $t \geq 3$ and

$$\psi(x) = \frac{ax+b}{cx+d}, \quad a, b, c, d \in \mathbb{Z}_{2^t}.$$

Assume, that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2.$$

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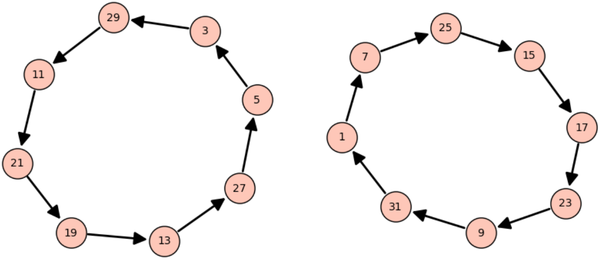
Then ψ defines a permutation on $\mathbb{Z}_{2^t}^*$.

If ψ is a permutation of $\mathbb{Z}_{2^t}^*$, then $(u_n) = (\psi^n(u))$ is purely periodic.

Its period length is $\tau = 2^k$ for some $0 \leq k \leq t-1$.

Examples for inversive generator



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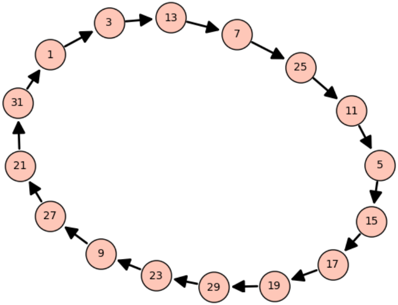

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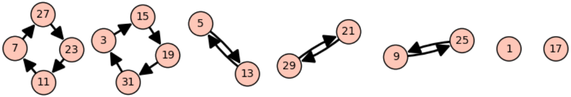

$$\psi(x) = \frac{x+4}{2x+3} \mod 2^5, \quad u_n = \psi(u_{n-1}), \quad n \geq 1.$$

Distribution for inversive generator

Assume, that ψ is a permutation of $\mathbb{Z}_{2^t}^*$ and let (u_n) be defined as

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Our goal is to study the discrepancy $D_N(u_n)$ of

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As usual, the main tool is to bound

$$\sum_{n=0}^{N-1} \exp\left(h\frac{2\pi i u_n}{2^t}\right), \quad \gcd(h,2) = 1.$$

Distribution for inversive generator

Niederreiter, Winterhof '05:

$$\sum_{n=0}^{N-1} \exp\left(h\frac{2\pi i u_n}{2^t}\right) \ll 2^{\frac{3}{4}t} N^{\frac{1}{2}} \tau^{-\frac{1}{2}}, \quad 1 \le N \le \tau,$$

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Can we do more?

Inversive generator, special cases

Proposition

Assume, that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has eigenvalue with multiplicity. Then*

$$u_n \equiv \frac{\alpha n + u_0}{\beta n + 1} \mod 2^t, \quad \textit{for } n \geq 0.$$

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Korolev '16:

$$\sum_{\substack{c \leq n < N+c \\ 2 \nmid n}} \exp\left(\frac{an^{-1} + bn}{2^t}\right) = o(N)$$

for $2^{c(t^{2/3})} \leq N \leq 2^{t/2}$.

Explicit inversive generator

Assume, that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is diagonalizable over $\mathbb{Z}_{2^t}$. Then

$$u_n \equiv \frac{\alpha}{g^n + \beta} + \gamma \mod 2^t, \quad n \geq 0, \quad \text{with } g, \alpha, \beta, \gamma \in \mathbb{Z}.$$

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Theorem (M., Shparlinski)

For odd α write $\tau = 2^{t-\nu+1}$.
Let $t > 16\nu$. Then for $2^{8\nu} < N \leq 2^{t/2}$,

$$\sum_{n=0}^{N-1} \exp\left(h \frac{2\pi i u_n}{2^t}\right) \ll N^{1-\varepsilon\left(\frac{\log N}{t}\right)^2}, \quad 2 \nmid h$$

for some $\varepsilon > 0$.

Explicit inversive generator – Discrepancy bound

Let $D_N(u_n)$ be the discrepancy of the sequence

$$u_0/2^t, \ldots, u_{N-1}/2^t \in [0,1).$$

Reminder:

$$D_N(u_n) = \sup_{I \subset [0,1)} \left| \frac{\#\{u_n \in I : 0 \leq n < N\}}{N} - |I| \right|.$$

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Corollary (M., Shparlinski)

For odd α write $\tau = 2^{t-\nu+1}$.
Let $t > 32\nu$. Then for $2^{8\nu} < N \le 2^{(\frac{1}{2}-\delta)t}$, with $0 < \delta < 1/2$,

$$D_N(u_n) \ll_\delta N^{-\varepsilon' \left(\frac{\log N}{t}\right)^2},$$

for some $\varepsilon' = \varepsilon'(\delta) > 0$.

Idea of the proof

- Dealing with exponential terms
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- Shift and average method:

$$\sum_{n=0}^{N-1} \exp\left(h\frac{2\pi i u_n}{2^t}\right) \approx \frac{1}{2^s}\sum_{n=0}^{N-1}\sum_{x=0}^{2^s-1} \exp\left(h\frac{2\pi i u_{n+x\cdot\tau_s}}{2^t}\right)$$

Thank you!