Multifractal analysis of the Brjuno function

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The result

Theorem (Jaffard-M, 2018)

The Brjuno function is multifractal.

Complex dynamics

Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ holomorphic with $f(0) = 0$, $f'(0) = \lambda \neq 0$ so that

$$f(z) = \lambda z + O(z^2).$$

Definition

f is linearizable at 0 if there exists $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ biholomorphic such that

$$h \circ f \circ h^{-1}(z) = \lambda z.$$

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Formally,

$$h(z) = z + \sum_{q \geqslant 2} \frac{b_q}{\lambda^q - \lambda} z^q,$$

with $b_q = P_q(a_2, \ldots, a_q, b_2, \ldots, b_{q-1})$ (if $f(z) = \sum_{k \geqslant 1} a_k z^k$).

Best denominators

Interesting case : $f'(0) = \lambda = \mathrm{e}^{2i\pi x}$ with $x \in]0, 1[\setminus\mathbb{Q}$. Then

$$h(z) = z + \frac{1}{\lambda} \sum_{q \geqslant 1} \frac{b_{q+1}}{e^{2i\pi qx} - 1} z^{q+1}.$$

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We introduce

$$\begin{cases} q_0 & = 1, \quad q_1 = 1 \text{ if } \frac{1}{2} < x < 1, \\ q_k & = \min\{q \in \mathbb{N} \,:\, \|qx\| < \|q_{k-1}x\|\} \quad (\|y\| = \mathrm{d}(y, \mathbb{Z})) \\ p_k & = \text{nearest integer to } q_k x. \end{cases}$$

$$\text{For all } k \geqslant 0, \quad \left| x - \frac{p_k}{q_k} \right| \leqslant \frac{1}{q_k q_{k+1}} \leqslant \frac{1}{q_k^2}.$$

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Irrationnality exponent of x :

$$\mu(x) := \sup\left\{\mu \geqslant 2 \,|\, \exists \text{ infinitely many } k, \left|x - \frac{p_k}{q_k}\right| \leqslant \frac{1}{q_k^{\mu}}\right\}.$$

Siegel, Brjuno, Yoccoz's results

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- Siegel (1942) : if $\mu(x) < \infty$, then f is linearizable at 0.
- Brjuno (1965) : if $\sum_{k \geqslant 0} \frac{\log q_{k+1}}{q_k} < \infty$, then f is linearizable at 0.
- Yoccoz (1987) : if $\sum_{k \geqslant 0} \frac{\log q_{k+1}}{q_k} = \infty$, then $z \mapsto e^{2i\pi x}(z - z^2/2)$ is not linearizable at 0.

The Brjuno function

For $x \in]0,1[\setminus\mathbb{Q}$, let

$$B(x) := \sum_{k\geqslant 0} |xq_{k-1} - p_{k-1}| \log\left(\frac{p_{k-1} - xq_{k-1}}{q_k x - p_k}\right) = \sum_{k\geqslant 0} xT(x)\dots T^{k-1}(x)\log(1/T^k(x)),$$

with $T : x \mapsto \frac{1}{x} \bmod 1$ (the Gauss map), and then extend B to $\mathbb{R}\setminus\mathbb{Q}$ by 1-periodicity.

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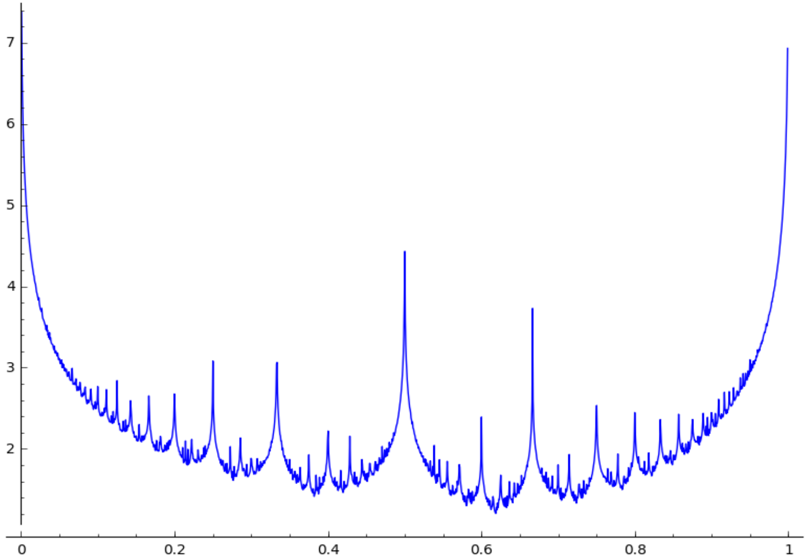
Properties :

1. $B(x) < \infty \Leftrightarrow \sum_{k \geqslant 0} \frac{\log q_{k+1}}{q_k} < \infty$ (Brjuno numbers)
2. $B \in L^1(0,1)$
3. $B \in BMO(\mathbb{R})$ (Marmi-Moussa-Yoccoz, 1997)
4. Functional equation on $]0,1[$: $B(x) - xB(T(x)) = \log(1/x)$



$x \mapsto B(x)$

Local behaviour of B

Theorem (Balazard-M, 2012)

Let $x_0 \in]0,1[$.

- *If* $B(x_0) < \infty$, *then* x_0 *is a Lebesgue point of* B *:*

$$\frac{1}{\rho}\int_{x_0-\rho}^{x_0+\rho} |B(x) - B(x_0)| dx \underset{\rho\to 0^+}{=} o(1) \quad ;$$

- *if* $B(x_0) = \infty$, *then*

$$\int_{x_0}^{x_0+\rho} B(x) dx \underset{\rho\to 0^+}{\to} +\infty \quad ;$$

- *if* $x_0 = p/q$, *then*

$$\int_{x_0}^{x_0+\rho} B(x) dx \underset{\rho\to 0^+}{\sim} \frac{\rho}{q}\log(1/\rho).$$

Main tools

- Functional equation of B ;
- The Gauss measure $\mu(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}$ on $[0,1]$ is invariant under the action of the Gauss map $x \mapsto 1/x \bmod 1$.

Hölder exponent

Let $f : \mathbb{R} \to \mathbb{R}$ locally bounded and $x_0 \in \mathbb{R}$.

Definition

- *For* $\beta \geqslant 0$,

$$f \in C^\beta(x_0) \text{ if } f(x) = P(x - x_0) + O(|x - x_0|^\beta), \quad (x \to x_0)$$

with P *a polynomial with* $\deg(P) < \lfloor \beta \rfloor$.

- $h_f(x_0) := \sup\{\beta \geqslant 0 \,:\, f \in C^\beta(x_0)\}$ *(Hölder-exponent of* f *at* x_0*)*

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- $f(x) = |x - x_0|^\alpha$ with $\alpha \notin 2\mathbb{N} \Rightarrow h_f(x_0) = \alpha$.

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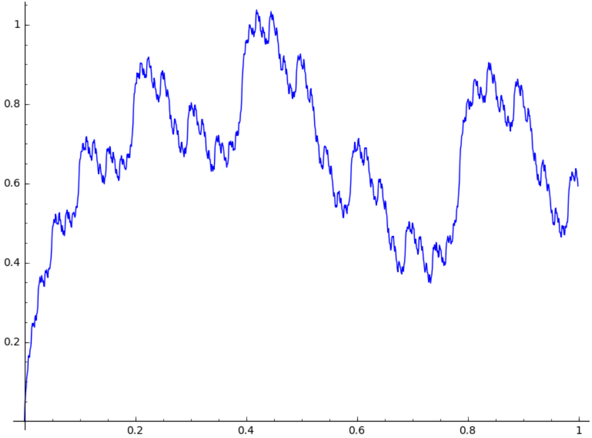
Iso-Hölder set : $H_f(\beta) := \{x_0 \in \mathbb{R} \; : \; h_f(x_0) = \beta\} \quad (\beta \in \mathbb{R})$

f is said to be multifractal if the image of $\beta \mapsto \dim(H_f(\beta))$ contains a non-trivial interval.

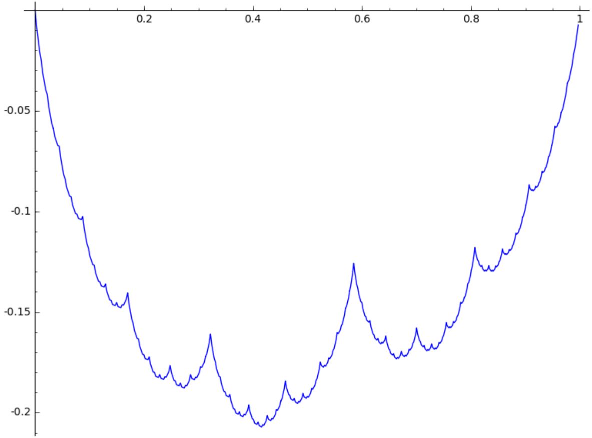
The Weierstrass function



$$x \mapsto W_a(x) = \sum_{n \geqslant 1} 2^{-na} \sin(2^n x) \quad (0 < a < 1)$$

$\forall x_0 \in \mathbb{R}, h_{W_a}(x_0) = a.$

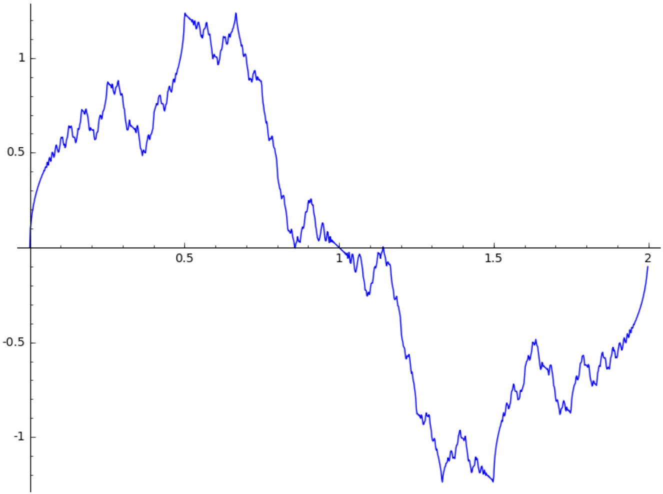
The Delange function



Graph of G defined by $\sum_{n \leqslant x} s_2(n) = \frac{x}{2} \log_2 x + x\, G(\{\log_2 x\})$.

$\forall x_0 \in \mathbb{R}, h_G(x_0) = 1.$

An example : the Riemann function



$$x \mapsto R(x) = \sum_{n \geqslant 1} \frac{\sin(\pi n^2 x)}{n^2}.$$

An example : the "non-differentiable" Riemann function

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Hardy (1916), Gerver (1970) :
R differentiable at $x \Leftrightarrow x = \frac{p}{q}$ with $(p, q) = 1$ and p, q both odd.

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After works by Itatsu (1981) and Duistermaat (1991) :

Theorem (Jaffard, 1996)

$$\dim(H_R(\beta)) = \begin{cases} 4\beta - 2 & \textit{si } \beta \in [\frac{1}{2}, \frac{3}{4}] \\ 0 & \textit{si } \beta = \frac{3}{2}, \\ -\infty & \textit{sinon.} \end{cases}$$

Other examples of multifractal functions : Jaffard (1997, 2004), Petrykiewicz (2014), Chamizo & Ubis (2014), Chamizo & Petrykiewicz & Ruiz-Cabello (2016), Seuret & Ubis (2017).

L^1 Hölder exponent

Recall : if $B(x_0) < \infty$, then $\frac{1}{\rho}\int_{x_0-\rho}^{x_0+\rho} |B(x) - B(x_0)| dx = o(1) \quad (\rho \to 0)$

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Let $f \in L^1_{loc}(\mathbb{R})$ and $x_0 \in \mathbb{R}$.

Definition

- *Let $\beta \geqslant -1$.*

$$f \in T^1(x_0) \text{ if } \frac{1}{2\rho} \int_{x_0-\rho}^{x_0+\rho} |f(x) - P(x - x_0)| dx = O(\rho^\beta) \quad (\rho \to 0),$$

with $P \in \mathbb{R}[X]$ and $deg(P) < \lfloor \beta \rfloor$.

- *L^1-Hölder exponent of f at x_0 :*

$$h_f^1(x_0) = \sup\{\beta \geqslant -1 \,:\, f \in T^1(x_0)\}.$$

For $\beta \in \mathbb{R}$, $\quad H_f^1(\beta) := \{x_0 \in \mathbb{R} \,:\, h_f^1(x_0) = \beta\}$.

L^1-exponent of B

For $x_0 \in \mathbb{R} \setminus \mathbb{Q}$,

$$\mu(x_0) := \sup \left\{ \mu \geqslant 2 \,|\, \exists \text{ infinitely many } k, \left| x - \frac{p_k}{q_k} \right| \leqslant \frac{1}{q_k^\mu} \right\}$$

Theorem (Jaffard-M, 2018)

Let $x_0 \in \mathbb{R}$. Then, $h_B^1(x_0) = \begin{cases} \frac{1}{\mu(x_0)} & \text{si } x_0 \in \mathbb{R} \setminus \mathbb{Q}, \\ 0 & \text{sinon.} \end{cases}$

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For almost all x_0, $\mu(x_0) = 2$.

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Jarnik's theorem (1930) : $\dim\{x_0 \,:\, \mu(x_0) = \mu\} = \frac{2}{\tau}$

Consequence : $\dim(H_B^1(\beta)) = \begin{cases} 2\beta & \text{if } \beta \in [0, 1/2], \\ 0 & \text{else.} \end{cases}$

Wavelet coefficient

Let $H \in L^1(\mathbb{R})$, with compact support, bounded by 1 and with $\int_{\mathbb{R}} H = 0$.

Wavelet family generated by H : $H_{a,b}(t) = H\Big(\dfrac{t-b}{a}\Big) \quad (a > 0, b \in \mathbb{R})$

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For $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, we compute :

$$C_f(a,b) = \frac{1}{a}\int_{\mathbb{R}} f(x) H_{a,b}(x)dx \quad \Big(= \frac{1}{a}\int_{\mathbb{R}} (f(x) - f(x_0)) H_{a,b}(x)dx \Big)$$

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Principle : if $\mathrm{supp}(H_{a,b}) \subset]x_0 - \rho, x_0 + \rho[$, then

$$h^1_f(x_0) > \beta \Longrightarrow |C_f(a,b)| \leqslant \frac{\rho^{1+\beta}}{a}. \tag{1}$$

Here we use the Haar-wavelet : $H = \mathbf{1}_{[0,1/2[} - \mathbf{1}_{[1/2,1[}$.

Global regularity of B

Theorem (Marmi-Moussa-Yoccoz, 1997)

$$B \in BMO(\mathbb{R}) : \sup_J \frac{1}{|J|} \int_J \left| B - \int_J B \right| < \infty$$

Let :

- $\lambda = e^{2i\pi x}$ with $x \in \mathbb{R} \setminus \mathbb{Q}$;
- $P_\lambda(z) = \lambda(z - z^2/2)$
- H_λ analytic at 0 such that $H_\lambda^{-1} \circ P_\lambda \circ H_\lambda = \lambda z$;
- $r(x)$ the radius of convergence of H_λ ;

Conjecture (Marmi-Moussa-Yoccoz, 1997)

$\Phi : x \mapsto B(x) + \log(r(x))$ *can be extended to a* $C^{1/2}$ *function.*

Buff-Chéritat (2006) : Φ can be extended to a continous function.