Uncertainty in Finite Affine Planes

Vsevolod F. Lev

The University of Haifa

Uniform Distribution Theory — CIRM, October 2018

Joint work with András Biró (Rényi Institute)

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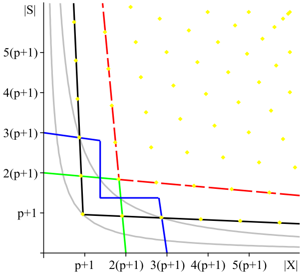

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What is Uncertainty About?

General Uncertainty Principle

A (nonzero) function and its Fourier transform cannot be both highly concentrated on small sets.

The framework:

- G — finite abelian group ($G = \mathbb{F}_p^2$ of particular interest);
- $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ — the dual group, $\chi \in \widehat{G}$ — characters;
- for $f\colon G \to \mathbb{C}$, the Fourier transform $\widehat{f}\colon \widehat{G} \to \mathbb{C}$ is defined by

$$\widehat{f}(\chi) := \frac{1}{|G|} \sum_{g \in G} f(g)\,\overline{\chi}(g), \quad \chi \in \widehat{G};$$

- $\operatorname{supp} f := \{g \in G\colon f(g) \neq 0\}$, and $\operatorname{supp} \widehat{f} := \{\chi \in \widehat{G}\colon \widehat{f}(\chi) \neq 0\}$.

The Basic Uncertainty Inequality

For any (nonzero) function $f\colon G \to \mathbb{C}$, we have

$$|\operatorname{supp} f|\,|\operatorname{supp} \widehat{f}| \geq |G|.$$

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Prime-order Groups: The Biró-Tao Theorem

$$|\operatorname{supp} f||\operatorname{supp} \widehat{f}| \geq |G|$$

For a subgroup $H \leq G$, let

$$H^{\perp} := \{\chi \in \widehat{G} \colon \chi|_H = 1\} \leq \widehat{G}.$$

The mapping $H \mapsto H^{\perp}$ establishes a bijection between the subgroups of G and those of $\widehat{G}$.

Since $\widehat{1_H} = \frac{1}{[G:H]} 1_{H^{\perp}}$, we have

$$|\operatorname{supp} 1_H||\operatorname{supp} \widehat{1_H}| = |H||H^{\perp}| = |G|,$$

showing that the Basic Uncertainty Inequality is sharp.

Theorem (Biró 1998, Tao 2005)

If G is cyclic of prime order, then in fact

$$|\operatorname{supp} f| + |\operatorname{supp} \widehat{f}| \geq |G| + 1.$$

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Meshulam's Generalization

Meshulam extended this to *arbitrary* finite abelian groups:

Theorem (Meshulam 2006)

Suppose that G is a finite abelian group, and $f\colon G \to \mathbb{C}$. If $d_1 < d_2$ are two consecutive divisors of $|G|$ with $d_1 \le |\operatorname{supp} f| \le d_2$, then

$$|\operatorname{supp} \widehat{f}| \ge \frac{|G|}{d_1 d_2}\,(d_1 + d_2 - |\operatorname{supp} f|).$$

- If, say, $|\operatorname{supp} f| = d_1$, then $|\operatorname{supp} \widehat{f}| \ge \frac{|G|}{d_1 d_2} \cdot d_2 = \frac{|G|}{|\operatorname{supp} f|}$; the very same estimate follows from the Basic Uncertainty Inequality. Meshulam improves over the Basic Uncertainty Inequality when $|\operatorname{supp} f|$ stays away from any divisor of $|G|$;
- If $p = |G|$ is prime, then we are forced to take $d_1 = 1$ and $d_2 = p$, to get $|\operatorname{supp} \widehat{f}| \ge 1 + p - |\operatorname{supp} f|$; this is the Biró-Tao theorem;
- Meshulam's proof uses induction, with the Biró-Tao theorem serving as a base case.

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The Three Pillars

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$$|\operatorname{supp} f||\operatorname{supp} \widehat{f}| \geq |G|.$$

Theorem (Biró 1998, Tao 2005)

$$|G| = p \quad \Rightarrow \quad |\operatorname{supp} f| + |\operatorname{supp} \widehat{f}| \geq p + 1.$$

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$$d_1 \leq |\operatorname{supp} f| \leq d_2 \quad \Rightarrow \quad |\operatorname{supp} \widehat{f}| \geq \frac{|G|}{d_1 d_2}(d_1 + d_2 - |\operatorname{supp} f|).$$

Meshulam's theorem shows that in $\mathbb{R}^2$, the points $(|\operatorname{supp} f|, |\operatorname{supp} \widehat{f}|)$ lie on or above the convex polygonal line through the points $(|H|, |G/H|)$, where H ranges over all subgroups of G. At the same time, the Basic Uncertainty Inequality merely states that the points $(|\operatorname{supp} f|, |\operatorname{supp} \widehat{f}|)$ lie on or above the hyperbola through the points $(|H|, |G|/|H|)$.

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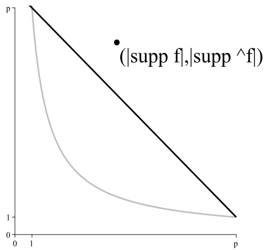
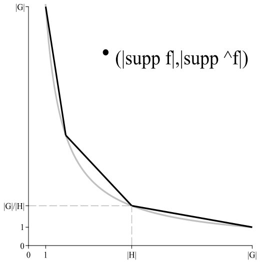
The Basic Inequality / Biró-Tao / Meshulam Visualized





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$$x + y = p + 1$$

$$xy = |G|$$
$$y = \frac{|G|}{d_1 d_2}(d_1 + d_2 - x)$$

Tao conjectured that the Basic Uncertainty Inequality can *always* be strengthened provided that $|\operatorname{supp} f|$ and $|\operatorname{supp} \widehat{f}|$ stay away from any divisor of $|G|$. Meshulam's Theorem confirms this conjecture.

In fact, it might be sufficient to assume that $\operatorname{supp} f$ and $\operatorname{supp} \widehat{f}$ "stay away from any coset of a subgroup of G".

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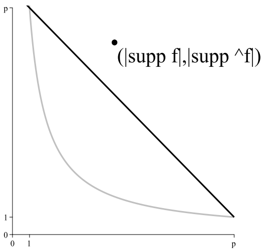

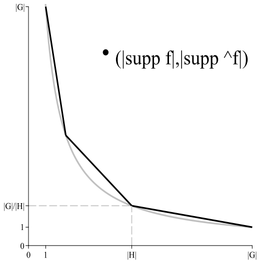

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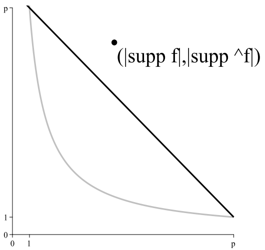

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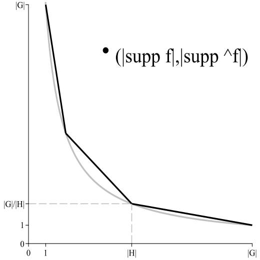

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Getting beyond Meshulam

From now on, $G = \mathbb{F}_p^2$.

Given a function $f\colon \mathbb{F}_p^2 \to \mathbb{C}$, we write $S := \operatorname{supp} f$ and $X := \operatorname{supp} \widehat{f}$.

Meshulam's Theorem for $G = \mathbb{F}_p^2$

For any nonzero function $f\colon \mathbb{F}_p^2 \to \mathbb{C}$, we have

$$\min\{|S|, |X|\} + \frac{1}{p} \max\{|S|, |X|\} \geq p + 1.$$

$xy = p^2$
— the Basic Uncertainty Inequality

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— Meshulam's estimate

- If $f = 1_H$, then $\widehat{f} = C1_{H^\perp}$ and $|S| = |X| = p$. Thus, to improve Meshulam's bound, one needs to take into account the *structure*.

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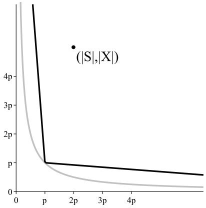

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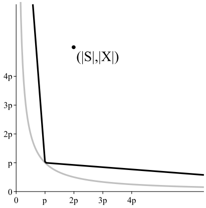

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The Conjecture

Meshulam's Theorem ($G = \mathbb{F}_p^2$, $S = \operatorname{supp} f$, $X = \operatorname{supp} \widehat{f}$)

For any nonzero function $f \colon \mathbb{F}_p^2 \to \mathbb{C}$, we have

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Conjecture

For any nonzero function $f \colon \mathbb{F}_p^2 \to \mathbb{C}$, and any $k \in [1, p]$, we have

$$\frac{1}{k} \min\{|S|, |X|\} + \frac{1}{p+1-k} \max\{|S|, |X|\} \ge p + 1,$$

unless at least one of the sets $S \subseteq \mathbb{F}_p^2$ and $X \subseteq \widehat{\mathbb{F}_p^2}$ is a dense subset of a union of a small number of proper cosets of the corresponding group.

- The case $k = 1$ of the conjecture is Meshulam's Theorem, the case $k = p$ follows from it;
- generally, for $k < p/2$, the "case k" implies the "case $p + 1 - k$".

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$$\frac{1}{k} \min\{|S|, |X|\} + \frac{1}{p+1-k} \max\{|S|, |X|\} \geq p + 1,$$

unless at least one of the sets $S \subseteq \mathbb{F}_p^2$ and $X \subseteq \widehat{\mathbb{F}_p^2}$ is a dense subset of a union of a small number of proper cosets of the corresponding group.

- The case $k = 1$ of the conjecture is Meshulam's Theorem, the case $k = p$ follows from it;
- generally, for $k < p/2$, the "case k" implies the "case $p + 1 - k$".

The Conjecture

Meshulam's Theorem ($G = \mathbb{F}_p^2$, $S = \operatorname{supp} f$, $X = \operatorname{supp} \widehat{f}$)

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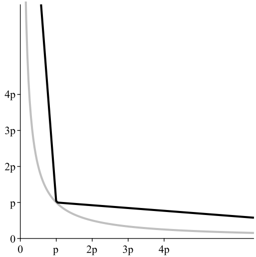
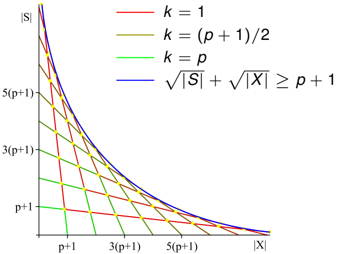
Visualizing the Conjecture

For a non-zero function $f\colon \mathbb{F}_p^2 \to \mathbb{C}$, and $k \in [1, p]$, we "normally" have

$$\frac{1}{k}\min\{|S|,|X|\} + \frac{1}{p+1-k}\max\{|S|,|X|\} \ge p+1.$$





Conjecture (restated)

For a non-zero function $f\colon \mathbb{F}_p^2 \to \mathbb{C}$, we "normally" have

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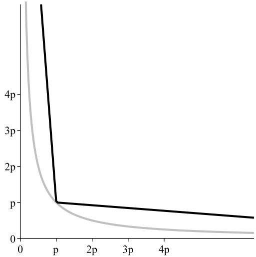

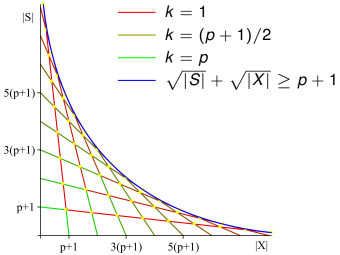

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Rational Functions: the Case $k = 2$

Theorem (establishing the case $k = 2$ for rational functions only)

If $f \colon \mathbb{F}_p^2 \to \mathbb{Q}$ is nonzero, then (writing $S := \operatorname{supp} f$ and $X := \operatorname{supp} \widehat{f}$),

$$\frac{1}{2}\min\{|S|,|X|\} + \frac{1}{p-1}\max\{|S|,|X|\} \ge p+1,$$

except if there is a nonzero, proper subgroup $H < \mathbb{F}_p^2$ such that f is constant on each H-coset (in which case $X = H^\perp$ or $X = H^\perp \setminus \{1\}$).

- If $H_1, H_2 < \mathbb{F}_p^2$ are (distinct) nonzero, proper subgroups, then for the function $f := 1_{H_1} - 1_{H_2}$ we have $|S| = |X| = 2(p-1)$.
- Hence, equality holds in this case, showing that the estimate is sharp and cannot be improved.

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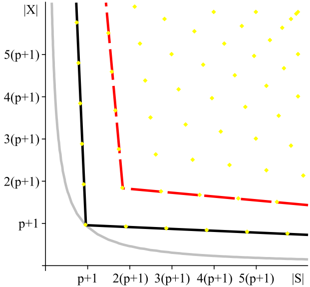

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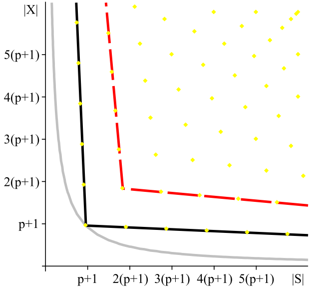

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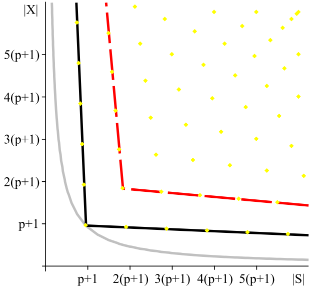

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Dropping the rationality assumption, we could prove the following:

Theorem

If $f\colon \mathbb{F}_p^2 \to \mathbb{C}$ is nonzero, then

$$\frac{1}{2}\min\{|S|, |X|\} + \frac{1}{\sqrt{p}}\max\{|S|, |X|\} \geq p,$$

except if the smallest of the sets S and X is contained in a coset of a nonzero, proper subgroup of the corresponding group ($\mathbb{F}_p^2$ or $\widehat{\mathbb{F}_p^2}$).

The Next (Easiest) Case: $k = p - 1$

Theorem

If $f \colon \mathbb{F}_p^2 \to \mathbb{C}$ is nonzero, then (writing $S := \operatorname{supp} f$ and $X := \operatorname{supp} \widehat{f}$),

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- If $f = 1_{g_1+H} - 1_{g_2+H}$, then $|S| = 2p$ and $|X| = p - 1$; hence, equality holds in this case, showing that the estimate is sharp.
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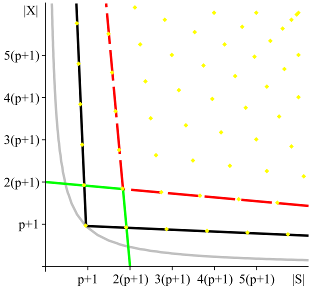

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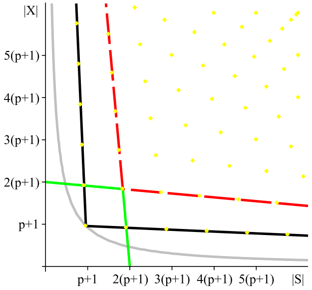

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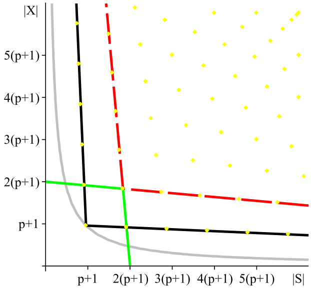

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The Case $k = p - 2$

Theorem (a partial result towards the case $k = p - 2$)

If $f \colon \mathbb{F}_p^2 \to \mathbb{C}$ is nonzero, then we have either

$$\frac{1}{p-2}\min\{|S|,|X|\} + \frac{1}{3}\max\{|S|,|X|\} \geq p+1,$$

or

$$\min\{|S|,|X|\} \geq \frac{3}{2}(p-1),$$

except if the smallest of S and X is "essentially" a coset, and the largest is a coset, or a union of two cosets of the orthogonal subgroup.

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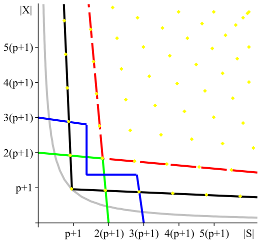

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A UDT Application: Counting Perfect Directions

Given a set $P \subseteq \mathbb{F}_p^2$ and a "weight function" $w\colon P \to \mathbb{R}$ (not vanishing identically), we say that a direction in $\mathbb{F}_p^2$ is *perfect* if every line in this direction gets its exact share of the total weight; that is, for every two lines ℓ_1, ℓ_2 in the direction in question, we have

$$\sum_{x \in P \cap \ell_1} w(x) = \sum_{x \in P \cap \ell_2} w(x).$$

If $|P| = 1$, there are no perfect directions. If $|P| = 2$ or $|P| = 3$, there is at most one perfect direction, for $|P| = 4$ there can be two. In general, how many perfect directions can there be?

Theorem

If $P \subseteq \mathbb{F}_p^2$, and $w\colon P \to \mathbb{R}$ does not vanish identically, then there are at most $\frac{1}{2}|P|$ perfect directions, unless there is a line ℓ entirely contained in P such that w is constant on ℓ, and vanishes outside of ℓ (in which case all, but one direction are perfect).

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Example

If $P = \ell_1 \cup \ell_2$ is a union of two parallel lines, and $w = c_1 1_{\ell_1} + c_2 1_{\ell_2}$ (that is, w is constant on each of these lines), then $|P| = 2p$ and there are $p = \frac{1}{2}|P|$ perfect directions.

Example

If $P = (\ell_1 \cup \ell_2) \setminus (\ell_1 \cap \ell_2)$ with ℓ_1 and ℓ_2 *not* parallel, and $w = 1_{\ell_1} - 1_{\ell_2}$, then $|P| = 2(p-1)$ and there are $p - 1 = \frac{1}{2}|P|$ perfect directions.

(There are more elaborate examples showing that $\frac{1}{2}|P|$ is sharp.)

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Sketch of the Proof

$P \subseteq \mathbb{F}_p^2$, $w\colon P \to \mathbb{R}$ $\Leftrightarrow$ there are at most $\frac{1}{2}|P|$ perfect directions

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We assume that w is defined on $\mathbb{F}_p^2$ (just let $w(x) = 0$ when $x \notin P$), and that $\operatorname{supp} w = P$ (by restricting P). WLOG, w is *rational-valued* (by simultaneously approximating $w(x),\ x \in P$).

Let $\mathcal{D}$ be the set of all $p+1$ directions in $\mathbb{F}_p^2$, each direction $\partial \in \mathcal{D}$ understood as a pencil of p parallel lines. Write $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$, where $\mathcal{D}^+$ is the set of all perfect directions.

Consider the decomposition

$$L_{\mathbb{Q}}(\mathbb{F}_p^2) = (\oplus_{\partial\in\mathcal{D}} V_\partial) \oplus V_0 :$$

- $L_{\mathbb{Q}}(\mathbb{F}_p^2)$ is the vector space of all rational-valued functions on $\mathbb{F}_p^2$;
- $V_\partial < L_{\mathbb{Q}}(\mathbb{F}_p^2)$ is the subspace of all zero-mean functions which are constant on every line $\ell \in \partial$;
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If ∂ is perfect, then w has zero projection onto V_∂!

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- $L_{\mathbb{Q}}(\mathbb{F}_p^2)$ is the vector space of all rational-valued functions on $\mathbb{F}_p^2$;
- $V_\partial < L_{\mathbb{Q}}(\mathbb{F}_p^2)$ is the subspace of all zero-mean functions which are constant on every line $\ell \in \partial$;
- $V_0 < L_{\mathbb{Q}}(\mathbb{F}_p^2)$ is the subspace of all constant functions.

If ∂ is perfect, then w has zero projection onto V_∂!

Sketch of the Proof (Concluded)

$P \subseteq \mathbb{F}_p^2$, $w\colon P \to \mathbb{R}$ $\Leftrightarrow$ there are at most $\frac{1}{2}|P|$ perfect directions

As a result,

$$w = \sum_{\partial \in \mathcal{D}^-} w_\partial + C, \quad w_\partial \in V_\partial.$$

But $w_\partial \in V_\partial$ implies that most of the Fourier coefficients of w_∂ vanish:

$$\operatorname{supp} \widehat{w_\partial} \subseteq H^\perp \setminus \{1\},$$

where $H < \mathbb{F}_p^2$ is the subgroup corresponding to the direction ∂. Hence,

$$|\operatorname{supp} \widehat{w}| \le (p-1)|\mathcal{D}^-| + 1.$$

Applying to w the estimate

$$\frac{1}{2}\,|\operatorname{supp} f| + \frac{1}{p-1}\,|\operatorname{supp} \widehat{f}| \ge p+1$$

from Slide 9, we conclude that, "normally",

$$p+1 \le \frac{1}{2}\,|P| + |\mathcal{D}^-| + \frac{1}{p-1} < \frac{1}{2}\,(|P|+1) + |\mathcal{D}^-|,$$

implying

$$|\mathcal{D}^+| = p+1-|\mathcal{D}^-| < \frac{1}{2}\,(|P|+1).$$

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Summary

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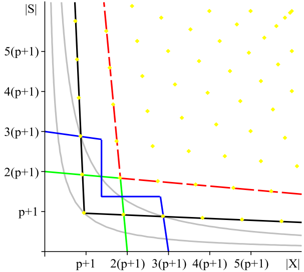

Thank you!