Automatic and q-multiplicative sequences through the lens of Higher order Fourier analysis

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6th International Conference on Uniform Distribution Theory
CIRM, 1 – 5 October 2018

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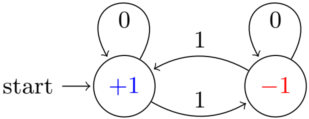

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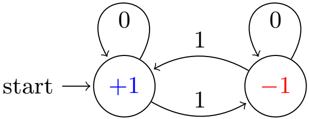

4. 2-Multiplicative sequence: $t(2^j) = -1$ for all j, and

$$t(n+m) = t(n)t(m)$$

if digits of n and m do not overlap, i.e., $2^i \mid n$ and $m < 2^i$ for some i.

Uniformity of Thue–Morse (1/3)

Question (Mauduit & Sarközy (1998))

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2. $\#\{n < N : t(n) = t(n+1)\} \simeq N/3 \neq N/2$. $\longrightarrow$ $t(n) = t(n+1)$ iff $2 \nmid \nu_2(n+1)$
3. $\#\{n < N : t(n) = t(n+1) = t(n+2)\} = 0$. $\longrightarrow$ in general: t is cube-free

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3. $\sup_{\alpha\in\mathbb{R}} \left| \mathop{\mathbb{E}}_{n<N} t(n)e(n\alpha) \right| = O(N^{-c})$ with $c > 0$. $\longrightarrow$ shorthand: $e(\theta) = e^{2\pi i\theta}$

Uniformity of Thue–Morse (2/3)

Gelfond problems

1. Thue-Morse does not correlate with the primes:

$$\#\{n < N \,:\, n \text{ is prime},\ t(n) = +1\} = \frac{1}{2}\pi(N) + O(N^{1-c}).$$

$\longrightarrow \pi(N) = \#$ primes $\leq N$; Prime Number Theorem: $\pi(N) \sim N/\log N$

Proved by Mauduit & Rivat (2010).

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2. If $p(x) \in \mathbb{R}[x]$, $p(\mathbb{N}) \subset \mathbb{N}$, then t does not correlate with the values of p:

$$\#\{n < N \ : \ t(p(n)) = +1\} = \frac{1}{2}N + O(N^{1-c}).$$

   Proved by Mauduit & Rivat (2009) for $p(n) = n^2$ (already known for $\deg p = 1$).

   Improved by Drmota, Mauduit & Rivat (2013): $t(n^2)$ is *strongly normal*, i.e.,

$$\#\left\{n < N \ : t((n+i)^2) = \epsilon_i \text{ for } 0 \leq i < k\right\} = \frac{1}{2^k}N + O(N^{1-c})$$

   for any $k \in \mathbb{N}$ and any $\epsilon_i \in \{+1, -1\}$ for $0 \leq i < k$.

   For $\deg p \geq 3$ — open problem!

Uniformity of Thue–Morse (3/3)

A problem that is not Gelfond's $\longrightarrow$ Drmota (2014)

3. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$; then

$$\#\{n < N : t(\lfloor n^\alpha \rfloor) = +1\} = \frac{1}{2}N + O(N^{1-c}).$$

Proved for $\alpha < 3/2$ by Müllner & Spiegelhofer (2017).
Moreover, for such α, $t(\lfloor n^\alpha \rfloor)$ is strongly normal.

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Question

Fix $k \in \mathbb{N}$. How many k-term arithmetic progressions $n,\ n+m,\ \ldots,\ n+(k-1)m$ contained in $\{0,\ldots,N-1\}$ are there such that $t(n+im) = 1$ for $0 \le i < k$?

More generally, is it the case that

$$\#\left\{(m,n)\ :\ n+im < N \text{ and } t(n+im) = \epsilon_i \text{ for } 0 \le i < k\right\} \simeq \frac{N}{(k-1)2^{k+1}}$$

for any $\epsilon_i \in \{+1,-1\}$ $(0 \le i < k)$?

Fourier analysis: first glance

Problem: Let $A \subset [N]$, $\#A = \alpha N$ and $k \in \mathbb{N}$. How many k-term arithmetic progressions in A? Is there at least one?

$\longrightarrow$ $[N] := \{0, 1, \ldots, N-1\}$; we identify $[N] \simeq \mathbb{Z}/N\mathbb{Z}$ and assume N is prime.

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Fourier expansion:

$$1_A(n) = \sum_{\xi < N} \hat{1}_A(\xi) e\left(\frac{\xi n}{N}\right), \text{ where } \hat{1}_A(\xi) = \mathop{\mathbb{E}}_{n < N} 1_A(n) e\left(\frac{-\xi n}{N}\right)$$

Note that $\hat{1}_A(0) = \alpha$.

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Motto: A is random $\Longleftrightarrow$ $\hat{1}_A(\xi)$ are small for $\xi \neq 0$.

Lemma

Suppose that $\left|\hat{1}_A(\xi)\right| < \varepsilon$ for all $\xi \neq 0$. Then

$$\#\left\{(n, m) \in [N]^2 \ : \ n, n+m, n+2m \in A\right\} = \frac{\alpha^3}{4} N^2 + O(\varepsilon N^2).$$

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Corollary: The number of 3-term APs in $\{n \in [N] \ : \ t(n) = +1\}$ is $\sim N^2/32$.

Higher order Fourier analysis (1/2)

Fact (Fourier analysis is not enough)

There exist $A \subset [N]$, $\#A = \alpha N$ such that $\hat{1}_A(\xi) \simeq 0$ for $\xi \neq 0$ but the number of 4-term APs in A is not $\simeq \alpha^4 N^2/6$ (like for a random set).

Example: $A = \{n \in [N] : 0 \leq \{n^2\sqrt{2}\} < \alpha\}$. $\longrightarrow$ $\{x\} = x - \lfloor x \rfloor$

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Definition (Gowers norm)

Fix $s \geq 1$. Let $f\colon [N] \to \mathbb{R}$. Then $\|f\|_{U^s[N]} \geq 0$ is defined by:

$$\|f\|_{U^s[N]}^{2^s} = \mathop{\mathbb{E}}_{\mathbf{n}} \prod_{\omega \in \{0,1\}^s} f\left(n_0 + \omega_1 n_1 + \ldots \omega_s n_s\right),$$

where the average is taken over all parallelepipeds in $[N]$, i.e., over all $\mathbf{n} = (n_0, \ldots, n_s) \in \mathbb{Z}^{s+1}$ such that $n_0 + \omega_1 n_1 + \ldots \omega_s n_s \in [N]$ for all $\omega \in \{0,1\}^s$.

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Facts:

1. $\|f\|_{U^s[N]}$ is well-defined for $s \geq 1$, i.e., the average on the RHS is ≥ 0
2. $\|f\|_{U^1[N]} = |\mathbb{E}_n f(n)|$ and $\|f\|_{U^2[N]} \doteq \|\hat{f}\|_{\ell^4}$ $\longrightarrow$ true in $\mathbb{Z}/N\mathbb{Z}$ rather than $[N]$
3. $\|f\|_{U^1[N]} \ll \|f\|_{U^2[N]} \ll \|f\|_{U^3[N]} \ll \ldots$
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If $p \in \mathbb{R}[x]$, $f(n) = e(p(n))$, $\deg p = s$ then $\|f\|_{U^s[N]} \simeq 0$ but $\|f\|_{U^{s+1}[N]} = 1$.
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Theorem (Generalised von Neumann Theorem)

Fix $s \geq 1$. If $A \subset [N]$, $\#A = \alpha N$ and $\|1_A - \alpha 1_{[N]}\|_{U^s[N]} \leq \varepsilon$, then A contains as many $(s+1)$-term APs as a random set of the same size, up to an error of size ε:

$$\#\{(n, m) \in [N]^2 \; : \; n, n+m, \ldots, n+sm \in A\} = \alpha^s N^2/2s + O(\varepsilon N^2).$$

Gowers uniform sequences

Let μ denote the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \dots p_k \text{ where } p_1, \dots, p_k \text{ are distinct primes,} \\ 0 & \text{if } n \text{ is divisible by a square.} \end{cases}$$

Recall that μ is *multiplicative*, meaning that $\mu(mn) = \mu(m)\mu(n)$ if $\gcd(m, n) = 1$.

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Theorem (Green & Tao (2008+2012))

Fix $s \geq 2$. The Möbius function is Gowers uniform of order s:

$$\|\mu\|_{U^s[N]} \to 0 \text{ as } N \to \infty.$$

Hence, the primes contain many arithmetic progressions of length $s + 1$.

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Theorem (Frantzikinakis & Host (2017))

Let ν be a (bounded) multiplicative function and $s \geq 2$. Then

$$\|\nu\|_{U^s[N]} \to 0 \textit{ as } N \to \infty \textit{ if and only if } \|\nu\|_{U^2[N]} \to 0 \textit{ as } N \to \infty.$$

Higher order Fourier analysis meets Thue–Morse

Recall: $t(n) = \begin{cases} +1 \text{ if the sum of binary digits of } n \text{ is even,} \\ -1 \text{ if the sum of binary digits of } n \text{ is odd.} \end{cases}$.

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Theorem (K.)

Fix $s \geq 1$. There exists $c = c_s > 0$ such that $\|t\|_{U^s[N]} \ll N^{-c}$.

Key ideas: Write a recursive formula for $\|t\|_{U^s[2^L]}^{2^s}$.

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Corollary

Fix $s \geq 1$ and let $c = c_s$ be as above. Then for any $\epsilon_i \in \{+1, -1\}$, $(0 \leq i \leq s)$

$$\#\{(n,m) \ : n + im < N \text{ and } t(n+im) = \epsilon_i \text{ for } 0 \leq i \leq s\} = \frac{N^2}{2^{s+2}s} + O(N^{2-c}).$$

In particular, the number of $(s+1)$-term arithmetic progressions contained in the set $\{n < N \ : \ t(n) = +1\}$ is $N^2/2^{s+2}s + O(N^{2-c})$.

Higher order Fourier analysis meets k-multiplicative sequences

Definition

Fix $k \geq 2$. A sequence $f \colon \mathbb{N} \to \mathbb{C}$ is *k-multiplicative* if

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Example

Recall that $t(n)$ is 2-multiplicative. More generally: Let

$$s_k(n) = \text{sum of digits of digits of } n \text{ in base } k.$$

Then $e(\alpha s_k(n))$ is k-multiplicative for any $\alpha \in \mathbb{R}$. $\longrightarrow e(\theta) = e^{2\pi i \theta}$

Higher order Fourier analysis meets k-multiplicative sequences

Definition

Fix $k \geq 2$. A sequence $f: \mathbb{N} \to \mathbb{C}$ is *k-multiplicative* if

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Theorem (Fan & K.)

Let f be a (bounded) k-multiplicative function and $s \geq 2$. Then

$$\|f\|_{U^s[N]} \to 0 \text{ as } N \to \infty \text{ if and only if } \|f\|_{U^2[N]} \to 0 \text{ as } N \to \infty.$$

Higher order Fourier analysis meets Rudin–Shapiro

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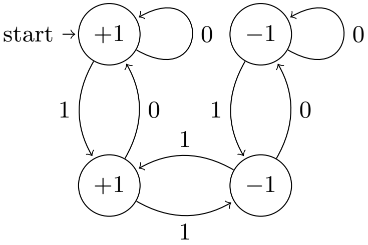

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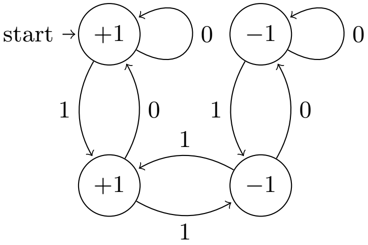

Theorem (K.)

Fix $s \geq 1$. There exists $c = c_s > 0$ such that $\|r\|_{U^s[N]} \ll N^{-c}$.

Automatic sequences

A sequence $f\colon \mathbb{N}_0 \to \Omega$ is k-automatic if and only if...

1. ...it is produced by a finite k-automaton $\mathcal{A} = (S, s_0, \delta, \tau)$.
   - S — a finite set of states, $s_0 \in S$ — initial state;
   - $\delta\colon S \times [k] \to S$ — transition function; uniquely extending to a map $\delta\colon S \times [k]^* \to S$ such that $\delta(s, uv) = \delta(\delta(s, u), v)$ for all $u, v \in [k]^*$;
     $\longrightarrow$ $[k]^*$ = words over the alphabet $[k] = \{0, \dots, k-1\}$
   - $\tau\colon S \to \Omega$ — output function.

   $\mathcal{A}$ computes the sequence $f_{\mathcal{A}}(n) := \tau(\delta(s, (n)_k)$. $\longrightarrow$ $(n)_k$ = digits of n in base k

2. ...it is given by a base k recurrence, i.e., the k-kernel $\mathcal{N}_k(f)$ is finite, where

$$\mathcal{N}_k(f) = \left\{ f(k^t n + r) \,:\, t \in \mathbb{N},\ 0 \le r < k^t \right\}.$$

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Motto: Automatic $\iff$ Computable by a finite device.

Higher order Fourier analysis meets automatic sequences

Question

Which among k-automatic sequences are Gowers uniform?
If a k-automatic sequence is uniform of order 2, must it be uniform of all orders?

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Example

The following sequences are 2-automatic and not Gowers uniform:

1. slowly varying sequences like $\lfloor \log_2(n) \rfloor \bmod 2$;
2. periodic sequences like $n \bmod 3$;
3. almost periodic sequences like $\nu_2(n) \bmod 2$.

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Conjecture (Byszewski, K. & Müllner)

Any k-automatic sequence has a decomposition $a = a_{\mathrm{str}} + a_{\mathrm{uni}}$, where

$$\|a_{\mathrm{uni}}\|_{U^s[N]} \ll N^{-c_s}$$

for any $s \geq 1$, and a_{str} is a "combination of sequences of the above type".

Obstructions to Gowers uniformity

Inverse Theorem for Gowers uniformity norms (Green, Tao & Ziegler (2012))

Fix $s \geq 1$. Let $f \colon [N] \to \mathbb{C}$ be a 1-bounded sequence. Then the following conditions are equivalent:

(1)
$$\|f\|_{U^s[N]} \geq \delta \text{ for certain } \delta > 0.$$

(2) there exists a 1-bounded $(s-1)$-step nilsequence φ with complexity $\leq C$ s. t.
$$\mathop{\mathbb{E}}_{n<N} f(n)\varphi(n) \geq \eta \text{ for certain } \eta > 0.$$

More precisely, if (1) holds for a given value of δ then there exist C and η, dependent on δ and s only (but not on N and f), such that (2) holds.
Conversely, if (2) holds for given values of C and η then there exists δ, dependent on C, η and s (but not on N and f), such that (1) holds.

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Motto: Obstructions to uniformity $\iff$ *Nilsequences (of bounded complexity).*

What are nilsequences?

Definition

A nilsequence $g\colon \mathbb{N} \to \mathbb{R}$ is a sequence such that there exists a nilsystem (X, T), a point $x_0 \in X$ and a continuous map $F\colon X \to \mathbb{R}$ such that $g(n) = F(T^n x_0)$.

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The generalised polynomial maps $\mathbb{Z} \to \mathbb{R}$ (denoted GP) are the smallest family such that $\mathbb{R}[x] \subset \mathrm{GP}$ and if $g, h \in \mathrm{GP}$ then also $g + h \in \mathrm{GP}$, $g \cdot h \in \mathrm{GP}$, $\lfloor g \rfloor \in \mathrm{GP}$.
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Theorem (Bergelson & Leibman (2007))

A bounded sequence $g\colon \mathbb{Z} \to \mathbb{R}$ is a generalised polynomial if and only if there exists a nilsystem (X, T), a point $x_0 \in X$ and a piecewise polynomial map $F\colon X \to \mathbb{R}$ such that $g(n) = F(T^n x_0)$. $\longrightarrow$ *Again, we over-simplify!*

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Motto: Obstructions to uniformity $\Longleftrightarrow$ Nilsequences $\Longleftrightarrow$ Generalised polynomials.

Can generalised polynomials be automatic?

Question

If f is both k-automatic and generalised polynomial, must f be ultimately periodic?

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Proposition (Allouche & Shallit (2003))

The sequence $f(n) = \lfloor \alpha n + \beta \rfloor \bmod m$ is automatic iff $\alpha \in \mathbb{Q}$. $(\alpha, \beta \in \mathbb{R},\ m \in \mathbb{N}_{\geq 2})$

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Ideas from Allouche & Shallit (implicit: circle rotation by α) combined with Bergelson & Leibman representation (rotations on nilmanifolds) lead to:

Proposition (Byszewski & K.)

If f is both k-automatic and generalised polynomial, then there exists a periodic sequence p and a set $Z \subset \mathbb{N}$ with $d^*(Z) = 0$ such that $f(n) = p(n)$ for all $n \in \mathbb{N} \setminus Z$.

$\longrightarrow$ d^* is the Banach density: $d^*(A) = \limsup_{N \to \infty} \max_M \dfrac{\#A \cap [M, M+N)}{N}$.

Sparse generalised polynomials

A *set* $A \subset \mathbb{N}$ is GP, k-automatic, etc. iff the sequence 1_A is GP, k-automatic, etc.

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If $A = \{a_1, a_2, \dots\} \subset \mathbb{N}$ and $\liminf_{i\to\infty} \frac{\log a_{i+1}}{\log a_i} > 1$, then A is GP.

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Fix $k \geq 2$. Then, one of the following holds:

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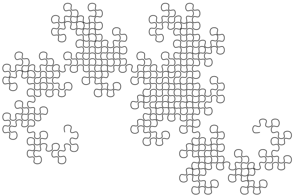
Thank you for your attention!



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