Uniform distribution mod 1, results and open problems

Jean-Marie De Koninck and Imre Kátai

6th International Conference on Uniform Distribution Theory

October 3, 2018
The plan

I. The Vienna conference of 2016

II. Old and recent results on regularly varying arithmetical functions

III. The Chowla conjecture and normal numbers

IV. Remarks on a theorem of Hedi Daboussi
Title: “New approaches in the construction of normal numbers”

available at:

www.jeanmariedekoninck.mat.ulaval.ca
Open problems formulated in 2016

1. Let $\mathcal{M}$ be the semi-group generated by the integers 2 and 3. Let $m_1 < m_2 < \cdots$ be the list of all the elements of $\mathcal{M}$. Is it possible to construct a real number $\alpha$ such that the sequence $(y_n)_{n \in \mathbb{N}}$, where $y_n = \{m_n \alpha\}$, is uniformly distributed in the interval $[0, 1)$?

2. Is it possible to construct a real number $\beta$ for which the corresponding sequence $(s_n)_{n \in \mathbb{N}}$, where $s_n = \{(\sqrt{2})^n \beta\}$, is uniformly distributed in the interval $[0, 1)$?
Conjectures formulated in 2016

3. Fix an integer \( q \geq 3 \) and let \( 1 = \ell_0 < \ell_1 < \cdots < \ell_{\varphi(q)-1} \) be the list of reduced residues modulo \( q \).

Let \( \varphi_q = \{ p \in \mathbb{P} : p \nmid q \} = \{ p_1, p_2, \ldots \} \).

For each \( p \in \varphi_q \), let \( h(p) = \nu \) if \( p \equiv \ell_\nu \pmod{q} \).

Let \( \alpha = 0.h(p_1)\,h(p_2)\,h(p_3)\ldots \) (\( \varphi(q) \)-ary expansion).

- **Conjecture 1**: \( \alpha \) is a \( \varphi(q) \)-ary normal number.

- **Conjecture 2**: \( \alpha \) is a \( \varphi(q) \)-ary normal number with weight \( 1/n \), that is, for every positive integer \( k \), given \( e_1 \ldots e_k \), an arbitrary block of \( k \) digits in \( \{0, 1, \ldots, \varphi(q) - 1\} \), we have

\[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n \leq N} \frac{1}{n} = \frac{1}{\varphi(q)^k}.
\]
I. The 2016 Vienna conference

Detailed account of our results is available in:

“Nineteen papers on normal numbers”

Accessible at:

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II. Old and recent results on arithmetic functions

If $f \in A$ and $
abla f(n) := f(n+1) - f(n) \to 0$ as $n \to \infty$, then $f(n) = c \log n$ (P. Erdős, 1946)

If $f \in A$ and $\sum_{n \leq x} |\nabla f(n)| \to 0$ as $x \to \infty$, then $f(n) = c \log n$ (conjectured by Erdős and proved by Kátai and Wirsing)

If $f \in M$ and $\nabla f(n) \to 0$ as $n \to \infty$, then $\lim_{n \to \infty} f(n) = 0$ or $f(n) = n^s$ (with $\Re(s) < 1$) (conjectured by Kátai and proved by Wirsing, Shao Pintsung and Tang Yuan Shang)
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If \( f \in \mathcal{A} \) and \( \Delta f(n) := f(n + 1) - f(n) \rightarrow 0 \) as \( n \rightarrow \infty \),
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- If \( f \in \mathcal{M} \) and \( \Delta f(n) \to 0 \) as \( n \to \infty \), then \( \lim_{n \to \infty} f(n) = 0 \) or \( f(n) = n^s \) (with \( \Re(s) < 1 \)) (conjectured by Kátai and proved by Wirsing, Shao Pintsung and Tang Yuan Shang)
Two conjectures of Kátai:

(a) If $f \in M$ and $\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |\Delta f(n)| = 0$, then either $\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0$ or $f(n) = ns$ (with $\Re(s) < 1$).

(b) If $f \in M$ and $\lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} |\Delta f(n)| = 0$, then either $\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0$ or $f(n) = ns$ (with $\Re(s) < 1$).

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(b) If \( f \in \mathcal{M} \) and \( \lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} \frac{|\Delta f(n)|}{n} = 0 \), then either \( \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0 \) or \( f(n) = n^s \) (with \( \Re(s) < 1 \)).
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Consider the set $M_1$ of multiplicative functions $f: \mathbb{N} \to T := \{z: |z| = 1\}$ and let $S(f)$ stand for the set of limit points of $\{f(n): n \in \mathbb{N}\}$.

Kátai's conjecture (Quebec Number Theory Conference, 1987): Let $f \in M_1$ be such that $S(f) = T$. Then, $S(\{f(n+1)f(n): n \in \mathbb{N}\}) = T$, except when $f(n) = n^\tau g(n)$ and $g_k(n) = 1$ for all $n \in \mathbb{N}$ for some positive integer $k$.

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As a consequence of Klurman’s result, the following could be proved:

(A) (Indlekofer, Kátai, Bui Minh Phong) If \( f \in M^* \) and 
\[
\sum_{n \leq x} |\Delta f(n)| \leq O(\log x),
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then either 
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\sum_{n \leq x} |f(n)| \leq O(\log x)
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or 
\[
f(n) = n\sigma + it
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for some \( \sigma \in (0,1) \).

(B) (Indlekofer, Kátai, Bui Minh Phong) If \( f \in M^* \) and 
\[
\limsup_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} |f(n + k) - f(n)| < \infty,
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where \( \chi \) is a Dirichlet character mod \( k \).

Remark. Perhaps similar theorems can be proved if one replaces 
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\Delta_k f(n) := f(n+k) - f(n)
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\[ k \in \{1, 2, 3\} \]

and assuming
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then either
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or
\[ f(n) = n \cdot s \cdot F(n) \]

with
\[ P(E)F(n) = 0. \]

(D) From the theorem of Klurman and Mangerel, one can prove the following.

Given an additive function \( f \) and \( \tau_1, \tau_2 \in \mathbb{R} \) such that \( \tau_2/\tau_1 \in \mathbb{R} \setminus \mathbb{Q} \). Then, if
\[ \lim_{x \to \infty} \frac{1}{\log x} \sum_{n \leq x} \| \tau_i \cdot \Delta f(n) \|_{n} = 0 \]

\((i = 1, 2)\), we have
\[ f(n) = c \cdot \log n. \]
(C) (Kátai and Bui Minh Phong) Given $k \in \{1, 2, 3\}$ and assuming that $\sum_{n \leq x} \frac{P(E)f(n)}{n} = O(\log x)$, then either $\sum_{n \leq x} \frac{|f(n)|}{n} = O(\log x)$ or $f(n) = n^sF(n)$ with $P(E)F(n) = 0$. 
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we have \( f(n) = c \log n \).
III. A problem related to the Chowla conjecture

According to the Chowla conjecture, if \( \lambda(n) \) stands for the Liouville function, 

\[
\sum_{n \leq x} \lambda(a_1 n + b_1) \cdots \lambda(a_k n + b_k) \to 0 \quad \text{as} \quad x \to \infty,
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provided that \( a_i b_j \neq b_i a_j \) for all \( 1 \leq i \neq j \leq k \).

Hence, setting \( \varepsilon_n = \lambda(n) + \frac{1}{2} \) for each \( n \in \mathbb{N} \) and assuming the Chowla conjecture, each number \( \xi_{k,\ell} := 0.\varepsilon_k + \ell \varepsilon_2 + \ell \varepsilon_3 + \cdots \) is a binary normal number for all integers \( k \geq 1 \) and \( \ell \geq 0 \).

**Problem.** Construct such a sequence \( \varepsilon_n \in \{0, 1\} \) for which the corresponding numbers \( \xi_{k,\ell} \) represent binary normal numbers.
III. A problem related to the Chowla conjecture

According to the Chowla conjecture, if $\lambda(n)$ stands for the Liouville function,

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Hence, setting $\varepsilon_n = \frac{\lambda(n)+1}{2}$ for each $n \in \mathbb{N}$ and assuming the Chowla conjecture, each number

$$\xi_{k,\ell} := 0.\varepsilon_{k+\ell}\varepsilon_{2k+\ell}\varepsilon_{3k+\ell} \cdots$$

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**Problem.** Construct such a sequence $\varepsilon_n \in \{0, 1\}$ for which the corresponding numbers $\xi_{k,\ell}$ represent binary normal numbers.
IV. Remarks on a theorem of Hedi Daboussi

Daboussi (1974) : Given any irrational number \( \alpha \) and any complex-valued multiplicative function \( f \) such that \( |f(n)| \leq 1 \), one has
\[
\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) e(\alpha n) = 0.
\]

Kátai : Let \( \tilde{\mathbb{P}} \) be a set of primes satisfying
\[
\sum_{p \in \tilde{\mathbb{P}}} \frac{1}{p} = +\infty
\]
and let \( f: \mathbb{N} \to \mathbb{C} \) and \( u: \mathbb{N} \to \mathbb{C} \) be two functions satisfying
\[
|f(n)| \leq 1 \quad \text{and} \quad |u(n)| \leq 1 \quad \text{for all} \quad n \in \mathbb{N}.
\]
If
\[
(1) \quad \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} u(p_1 n) u(p_2 n) = 0 \quad \text{for every} \quad p_1 \neq p_2 \in \tilde{\mathbb{P}},
\]
\[
(2) \quad f(pm) = f(p) f(m) \quad \text{for all} \quad p \in \tilde{\mathbb{P}} \quad \text{and} \quad m \in \mathbb{N},
\]
then,
\[
\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) u(n) = 0.
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Daboussi (1974) : Given any irrational number $\alpha$ and any complex-valued multiplicative function $f$ such that $|f(n)| \leq 1$, one has

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Kátai : Let $\tilde{\wp}$ be a set of primes satisfying $\sum_{p \in \tilde{\wp}} 1/p = +\infty$ and let $f : \mathbb{N} \to \mathbb{C}$ and $u : \mathbb{N} \to \mathbb{C}$ be two functions satisfying $|f(n)| \leq 1$ and $|u(n)| \leq 1$ for all $n \in \mathbb{N}$. If

1. $\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} u(p_1 n)u(p_2 n) = 0$ for every $p_1 \neq p_2 \in \tilde{\wp}$,

2. $f(pm) = f(p)f(m)$ for all $p \in \tilde{\wp}$ and $m \in \mathbb{N}$,

then, $\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)u(n) = 0$. 
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and let $f : \mathbb{N} \to \mathbb{C}$ and $u : \mathbb{N} \to \mathbb{C}$ be two functions satisfying

$$|f(n)| \leq 1 \text{ and } |u(n)| \leq 1 \quad \forall n \in \mathbb{N}.$$ 

If

(1) \( \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} u(p_1^n) u(p_2^n) = 0 \) for every \( p_1 \neq p_2 \in \tilde{\mathcal{P}} \),

(2) \( f(pm) = f(p)f(m) \) for all \( p \in \tilde{\mathcal{P}} \) and \( m \in \mathbb{N} \),

then,

(∗) \( \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) u(n) = 0 \).

Remark. More is true. Let $\tilde{\mathcal{P}}$ and $u$ be as above. Then (∗) holds uniformly for every $f$ satisfying the above condition (2) as well as the condition

$$|f(n)| \leq 1 \quad \forall n \in \mathbb{N}.$$
IV. Remarks on a theorem of Hedi Daboussi

Kátai : Let \( \tilde{\mathcal{P}} \) be a set of primes satisfying \( \sum_{p \in \tilde{\mathcal{P}}} \frac{1}{p} = +\infty \) and let \( f : \mathbb{N} \to \mathbb{C} \) and \( u : \mathbb{N} \to \mathbb{C} \) be two functions satisfying \( |f(n)| \leq 1 \) and \( |u(n)| \leq 1 \) \( \forall n \in \mathbb{N} \). If

1. \( \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} u(p_1 n) = 0 \) for every \( p_1 \neq p_2 \in \tilde{\mathcal{P}} \),

2. \( f(pm) = f(p)f(m) \) for all \( p \in \tilde{\mathcal{P}} \) and \( m \in \mathbb{N} \),

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\((*) \quad \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)u(n) = 0.\)
Kátai: Let $\tilde{\wp}$ be a set of primes satisfying $\sum_{p \in \tilde{\wp}} 1/p = +\infty$ and let $f : \mathbb{N} \to \mathbb{C}$ and $u : \mathbb{N} \to \mathbb{C}$ be two functions satisfying $|f(n)| \leq 1$ and $|u(n)| \leq 1 \ \forall n \in \mathbb{N}$. If

\begin{enumerate}
\item \[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} u(p_1 n)u(p_2 n) = 0 \text{ for every } p_1 \neq p_2 \in \tilde{\wp}, \]
\item \[ f(pm) = f(p)f(m) \text{ for all } p \in \tilde{\wp} \text{ and } m \in \mathbb{N}, \]
\end{enumerate}

then,

\[ (*) \quad \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)u(n) = 0. \]

**Remark.** More is true. Let $\tilde{\wp}$ and $u$ be as above. Then $(*)$ holds uniformly for every $f$ satisfying the above condition (2) as well as the condition $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. 

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Uniform distribution mod 1, results and open problems
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IV. Remarks on a theorem of Hedi Daboussi

Let $\mathcal{P}^*$ be a set of primes satisfying

$$\sum_{p \in \mathcal{P}^*} \frac{1}{p} \leq \tau x + O\left(\frac{x}{\log A x}\right)$$

for some positive $\tau$ and $A$. Further set

$$N(\mathcal{P}^*) := \{n \in \mathbb{N} : p | n \Rightarrow p \in \mathcal{P}^*\}$$

and $N(\mathcal{P}^*)(x) := \#\{n \leq x : n \in N(\mathcal{P}^*)\}$. Let $\mathcal{P}^{**} \subset \mathcal{P}^*$ such that

$$\sum_{p \in \mathcal{P}^{**}} \frac{1}{p} = +\infty.$$
Let $\mathcal{P}^*$ be a set of primes satisfying
$$\sum_{p \leq x, p \in \mathcal{P}^*} \log p = \tau x + O\left(x / \log^A x\right)$$
for some positive $\tau$ and $A$. Further set $\mathcal{N}(\mathcal{P}^*) := \{ n \in \mathbb{N} : p \mid n \implies p \in \mathcal{P}^* \}$ and $N_{\mathcal{P}^*}(x) := \#\{ n \leq x : n \in \mathcal{N}(\mathcal{P}^*) \}$. Let $\mathcal{P}^{**} \subset \mathcal{P}^*$ be such that
$$\sum_{p \in \mathcal{P}^{**}} 1/p = +\infty.$$ We can prove the following.
Let $\mathcal{P}^*$ be a set of primes satisfying
\[ \sum_{p \leq x, p \in \mathcal{P}^*} \log p = \tau x + O(x/\log^A x) \]
for some positive $\tau$ and $A$.

Further set
\[ N(\mathcal{P}^*) := \{ n \in \mathbb{N} : p \mid n \Rightarrow p \in \mathcal{P}^* \} \]
and
\[ N_{\mathcal{P}^*}(x) := \# \{ n \leq x : n \in N(\mathcal{P}^*) \} . \]

Let $\mathcal{P}^{**} \subset \mathcal{P}^*$ be such that
\[ \sum_{p \in \mathcal{P}^{**}} 1/p = +\infty . \]
We can prove the following.

Given $f : N(\mathcal{P}^*) \rightarrow \mathbb{C}$ satisfying $f(pm) = f(p)f(m) \ \forall p \in \mathcal{P}^{**}$ and $m \in N(\mathcal{P}^*)$ and such that $|f(n)| \leq 1 \ \forall n \in N(\mathcal{P}^*)$, then, for all functions $u : N(\mathcal{P}^*) \rightarrow \mathbb{C}$ such that $|u(n)| \leq 1 \ \forall n \in N(\mathcal{P}^*)$ and such that
\[ \lim_{x \to \infty} \frac{1}{x} \sum_{m \in N(\mathcal{P}^*)} u(p_1 m)u(p_2 m) = 0 \quad \text{for all } p_1 \neq p_2 \in \mathcal{P}^{**} , \]
we have
\[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \in N(\mathcal{P}^*)} f(n)u(n) = 0 . \]
Open problem

Is it true or not that, for every irrational number $\alpha$,

$$\lim_{x \to \infty} \frac{1}{N_{\varphi^*}(x)} \sum_{\substack{n < x \\ n \in \mathcal{N}(\varphi^*)}} e(n\alpha) = 0$$
Food for thought...

Thank you!

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