Uniform distribution mod 1, results and open problems

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6th International Conference on Uniform Distribution Theory

October 3, 2018

The plan

I. The Vienna conference of 2016

II. Old and recent results on regularly varying arithmetical functions

III. The Chowla conjecture and normal numbers

IV. Remarks on a theorem of Hedi Daboussi

I. The 2016 Vienna conference

Title : "New approaches in the construction of normal numbers"

available at :

`www.jeanmariedekoninck.mat.ulaval.ca`

Open problems formulated in 2016

1. Let $\mathcal{M}$ be the semi-group generated by the integers 2 and 3. Let $m_1 < m_2 < \cdots$ be the list of all the elements of $\mathcal{M}$. Is it possible to construct a real number α such that the sequence $(y_n)_{n\in\mathbb{N}}$, where $y_n = \{m_n\alpha\}$, is uniformly distributed in the interval $[0,1)$?

2. Is it possible to construct a real number β for which the corresponding sequence $(s_n)_{n\in\mathbb{N}}$, where $s_n = \{(\sqrt{2})^n\beta\}$, is uniformly distributed in the interval $[0,1)$?

Conjectures formulated in 2016

3. Fix an integer $q \geq 3$ and let $1 = \ell_0 < \ell_1 < \cdots < \ell_{\varphi(q)-1}$ be the list of reduced residues modulo q.
Let $\wp_q = \{p \in \wp : p \nmid q\} = \{p_1, p_2, \ldots\}$.
For each $p \in \wp_q$, let $h(p) = \nu$ if $p \equiv \ell_\nu \pmod{q}$.
Let $\alpha = 0.h(p_1)\,h(p_2)\,h(p_3)\ldots$ $\qquad$ ($\varphi(q)$-ary expansion).

- `Conjecture 1` : α is a $\varphi(q)$-ary normal number.
- `Conjecture 2` : α is a $\varphi(q)$-ary normal number with weight $1/n$, that is, for every positive integer k, given $e_1 \ldots e_k$, an arbitrary block of k digits in $\{0, 1, \ldots, \varphi(q) - 1\}$, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{\substack{n \leq N \\ h(p_{n+1})\ldots h(p_{n+k}) = e_1 \ldots e_k}} \frac{1}{n} = \frac{1}{\varphi(q)^k}.$$

I. The 2016 Vienna conference

Detailed account of our results is available in :

"Nineteen papers on normal numbers"

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II. Old and recent results on arithmetic functions

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- If $f \in \mathcal{A}$ and $\frac{1}{x}\sum_{n \le x} |\Delta f(n)| \to 0$ as $x \to \infty$, then $f(n) = c \log n$ (conjectured by Erdős and proved by Kátai and Wirsing)

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- If $f \in \mathcal{M}$ and $\Delta f(n) \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} f(n) = 0$ or $f(n) = n^s$ (with $\Re(s) < 1$) (conjectured by Kátai and proved by Wirsing, Shao Pintsung and Tang Yuan Shang)

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(b) If $f \in \mathcal{M}$ and $\lim_{x\to\infty} \frac{1}{\log x}\sum_{n\le x} \frac{|\Delta f(n)|}{n} = 0$, then either $\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} |f(n)| = 0$ or $f(n) = n^s$ (with $\Re(s) < 1$)

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In 2017, Oleksiy Klurman proved the two conjectures (a) and (b).

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Consider the set $\mathcal{M}_1$ of multiplicative functions $f : \mathbb{N} \to T := \{z : |z| = 1\}$ and let $S(f)$ stand for the set of limit points of $\{f(n) : n \in \mathbb{N}\}$.

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Kátai's conjecture (Quebec Number Theory Conference, 1987) :

Let $f \in \mathcal{M}_1$ be such that $S(f) = T$. Then,

$$S(\{f(n+1)\overline{f(n)} : n \in \mathbb{N}\}) = T,$$

except when $f(n) = n^{i\tau} g(n)$ and $g^k(n) = 1$ for all $n \in \mathbb{N}$ for some positive integer k.

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(B) (Indlekofer, Kátai, Bui Minh Phong) If $f \in \mathcal{M}^*$ and $\limsup_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|f(n+k)-f(n)|}{n} < \infty$, then either $\sum_{n\le x} \frac{|f(n)|}{n} = O(\log x)$ or $f(n) = n^{\sigma+it}\chi$, where χ is a Dirichlet character mod k.

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Remark. Perhaps similar theorems can be proved if one replaces $\Delta_k f(n) := f(n+k) - f(n)$ by $P(E) f(n) := a_0 f(n) + \cdots + a_k f(n+k)$.

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(C) (Kátai and Bui Minh Phong) Given $k \in \{1, 2, 3\}$ and assuming that $\sum_{n \le x} \frac{P(E)f(n)|}{n} = O(\log x)$, then either $\sum_{n \le x} \frac{|f(n)|}{n} = O(\log x)$ or $f(n) = n^s F(n)$ with $P(E)F(n) = 0$.

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(D) From the theorem of Klurman and Mangerel, one can prove the following.

Given an additive function f and $\tau_1, \tau_2 \in \mathbb{R}$ such that $\tau_2/\tau_1 \in \mathbb{R} \setminus \mathbb{Q}$. Then, if

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{n \le x} \frac{\|\tau_i \cdot \Delta f(n)\|}{n} = 0 \qquad (i = 1, 2),$$

we have $f(n) = c \log n$.

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According to the Chowla conjecture, if $\lambda(n)$ stands for the Liouville function,

$$\frac{1}{x}\sum_{n\le x}\lambda(a_1 n+b_1)\cdots\lambda(a_k n+b_k)\to 0 \text{ as } x\to\infty,$$

provided that $a_i b_j \neq b_i a_j$ for all $1\le i\neq j\le k$.

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Hence, setting $\varepsilon_n=\frac{\lambda(n)+1}{2}$ for each $n\in\mathbb{N}$ and assuming the Chowla conjecture, each number

$$\xi_{k,\ell}:=0.\varepsilon_{k+\ell}\varepsilon_{2k+\ell}\varepsilon_{3k+\ell}\ldots$$

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Problem. Construct such a sequence $\varepsilon_n\in\{0,1\}$ for which the corresponding numbers $\xi_{k,\ell}$ represent binary normal numbers.

IV. Remarks on a theorem of Hedi Daboussi

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Daboussi (1974) : Given any irrational number α and any complex-valued multiplicative function f such that $|f(n)| \leq 1$, one has

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Kátai : Let $\widetilde{\wp}$ be a set of primes satisfying $\sum_{p\in\widetilde{\wp}} 1/p = +\infty$ and let $f : \mathbb{N} \to \mathbb{C}$ and $u : \mathbb{N} \to \mathbb{C}$ be two functions satisfying $|f(n)| \le 1$ and $|u(n)| \le 1$ for all $n \in \mathbb{N}$. If

(1) $\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} u(p_1 n)\overline{u(p_2 n)} = 0$ for every $p_1 \ne p_2 \in \widetilde{\wp}$,

(2) $f(pm) = f(p)f(m)$ for all $p \in \widetilde{\wp}$ and $m \in \mathbb{N}$,

then, $\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)u(n) = 0$.

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Remark. More is true. Let $\widetilde{\wp}$ and u be as above. Then $(*)$ holds uniformly for every f satisfying the above condition (2) as well as the condition $|f(n)| \leq 1$ for all $n \in \mathbb{N}$.

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Let $\wp^*$ be a set of primes satisfying $\sum_{p \le x, p \in \wp^*} \log p = \tau x + O(x/\log^A x)$ for some positive τ and A. Further set $\mathcal{N}(\wp^*) := \{n \in \mathbb{N} : p \mid n \Longrightarrow p \in \wp^*\}$ and $N_{\wp^*}(x) := \#\{n \le x : n \in \mathcal{N}(\wp^*)\}$. Let $\wp^{**} \subset \wp^*$ be such that $\sum_{p \in \wp^{**}} 1/p = +\infty$. We can prove the following.

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Given $f : \mathcal{N}(\wp^*) \to \mathbb{C}$ *satisfying* $f(pm) = f(p)f(m)$ $\forall p \in \wp^{**}$ *and* $m \in \mathcal{N}(\wp^*)$ *and such that* $|f(n)| \le 1$ $\forall n \in \mathcal{N}(\wp^*)$*, then, for all functions* $u : \mathcal{N}(\wp^*) \to \mathbb{C}$ *such that* $|u(n)| \le 1$ $\forall n \in \mathcal{N}(\wp^*)$ *and such that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{m\le x\\ m\in\mathcal{N}(\wp^*)}} u(p_1 m)\overline{u(p_2 m)} = 0 \quad \textit{for all } p_1 \ne p_2 \in \wp^{**},$$

we have $\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x\\ n\in\mathcal{N}(\wp^*)}} f(n)u(n) = 0.$

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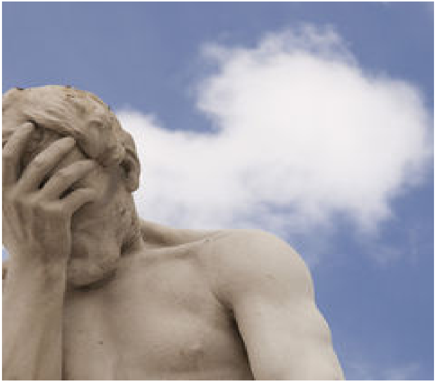
Open problem

Is it true or not that, for every irrational number α*,*

$$\lim_{x\to\infty} \frac{1}{N_{\wp^*}(x)} \sum_{\substack{n\le x \\ n\in\mathcal{N}(\wp^*)}} e(n\alpha) = 0 \quad ?$$

Food for thought. . .



Thank you !

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