On BRS and S-NBRS for Sequences $(\{a_n\alpha\})_{n\geq 1}$

UDT 2018

Lisa Kaltenböck
02.10.2018
joint work with Gerhard Larcher

discrepancy function

Definition

For a sequence $(x_n)_{n\geq 1}$ in $[0,1)$ and any subinterval $[a,b) \subseteq [0,1)$, let

$$D_N([a,b)) = \frac{\#\{1 \leq n \leq N : x_n \in [a,b)\}}{N} - (b-a)$$

be the (N-th) discrepancy function of $[a,b)$.

Then, the (N-th) discrepancy of (x_n) is given by

$$D_N = \sup_{[a,b)} |D_N([a,b))|.$$

bounds for the discrepancy

Theorem (Schmidt, 1972)

There exists a constant $c > 0$ such that for any infinite sequence $(x_n)_{n\geq 1}$ in $[0,1)$ it holds that

$$D_N \geq c\frac{\log N}{N}$$

for infinitely many N.

bounds for the discrepancy function

Theorem (Tijdeman, Wagner, 1980)

For every sequence $(x_n)_{n\geq 1}$ *we have for almost all* $b \in [0,1)$ *that*

$$|D_N([0,b))| \geq \frac{1}{400}\frac{\log N}{N}$$

for infinitely many N*.*

bounded remainder sets

Definition

Let $(x_n)_{n\geq 1}$ be a sequence in $[0,1)$ and let $[a,b) \subseteq [0,1)$. $[a,b)$ is called bounded remainder set (BRS) if there exists some constant c such that

$$|D_N([a,b))| \leq c\frac{1}{N}$$

for all $N \in \mathbb{N}$.

For any sequence there are only countably many possible values for the lengths of BRS. (Schmidt, 1972)

BRS for $(\{n\alpha\})_{n\geq 1}$

Theorem (Kesten, 1966)

An interval $[a,b) \subseteq [0,1)$ is a BRS for the sequence $(\{n\alpha\})_{n\geq 1}$ if and only if

$$b - a = \{j\alpha\}$$

for some $j \in \mathbb{Z}$.

" $\Leftarrow$ ": Hecke (1921), Ostrowski (1930)

BRS for $(\{n\alpha\})_{n\geq 1}$

Theorem (Kesten, 1966)

An interval $[a,b) \subseteq [0,1)$ *is a BRS for the sequence* $(\{n\alpha\})_{n\geq 1}$ *if and only if*

$$b - a = \{j\alpha\}$$

for some $j \in \mathbb{Z}$.

" $\Leftarrow$ ": Hecke (1921), Ostrowski (1930)

- sequences of the form $(\{a_n\alpha\})_{n\geq 1}$?

$a_n = 2^n$

The sequence $(\{2^n\alpha\})_{n\geq 1}$ is uniformly distributed modulo 1 if and only if $\alpha = 0, \alpha_1\alpha_2 \ldots_2$ is normal to base 2.

$a_n = 2^n$

The sequence $(\{2^n\alpha\})_{n\geq 1}$ is uniformly distributed modulo 1 if and only if $\alpha = 0, \alpha_1\alpha_2\ldots_2$ is normal to base 2.

$$\{2^n\alpha\} = \sum_{i=1}^{\infty} \alpha_{n+i}\frac{1}{2^i}$$

$a_n = 2^n$

The sequence $(\{2^n\alpha\})_{n\geq 1}$ is uniformly distributed modulo 1 if and only if $\alpha = 0, \alpha_1\alpha_2 \ldots_2$ is normal to base 2.

$$\{2^n\alpha\} = \sum_{i=1}^{\infty} \alpha_{n+i}\frac{1}{2^i}$$

$\Rightarrow$ arbitrarily many successive elements of $(\{2^n\alpha\})$ are contained in $[0, b)$ or not contained in $[a, b), a \neq 0$

$a_n = 2^n$

The sequence $(\{2^n\alpha\})_{n\geq 1}$ is uniformly distributed modulo 1 if and only if $\alpha = 0,\alpha_1\alpha_2\ldots_2$ is normal to base 2.

$$\{2^n\alpha\} = \sum_{i=1}^{\infty} \alpha_{n+i}\frac{1}{2^i}$$

$\Rightarrow$ arbitrarily many successive elements of $(\{2^n\alpha\})$ are contained in $[0,b)$ or not contained in $[a,b), a \neq 0$

$\Rightarrow$ $(\{2^n\alpha\})$ does not have any non-trivial BRS

$a_n = \beta^n$

More general, let $\beta > 1$ be a Pisot number and let $\alpha < 1$,

$$\alpha = \sum_{i=1}^{\infty} \alpha_i \beta^{-i},$$

where $\alpha_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ and such that $\sum_{i>j} \alpha_i \beta^{-i} < \beta^{-j}$ for all $j \geq 0$. Then

$\boldsymbol{a_n = \beta^n}$

More general, let $\beta > 1$ be a Pisot number and let $\alpha < 1$,

$$\alpha = \sum_{i=1}^{\infty} \alpha_i \beta^{-i},$$

where $\alpha_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ and such that $\sum_{i>j} \alpha_i \beta^{-i} < \beta^{-j}$ for all $j \geq 0$. Then

$$\sum_{n \geq 1} \|\beta^n\| < c,$$

for some constant c and

$\boldsymbol{a_n = \beta^n}$

More general, let $\beta > 1$ be a Pisot number and let $\alpha < 1$,

$$\alpha = \sum_{i=1}^{\infty} \alpha_i \beta^{-i},$$

where $\alpha_i \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ and such that $\sum_{i>j} \alpha_i \beta^{-i} < \beta^{-j}$ for all $j \geq 0$. Then

$$\sum_{n \geq 1} \|\beta^n\| < c,$$

for some constant c and
α has normal β-expansion $\Rightarrow (\{\beta^n \alpha\})_{n \geq 1}$ is u.d.

$a_n = \beta^n$

For any interval $[a, b) \subset [0, 1)$ with $a \neq 0$ and $b \neq 1$ there exist arbitrarily many successive elements of $(\{\beta^n \alpha\})_{n \geq 1}$ which are not contained in $[a, b)$.

Theorem (K., Larcher)

Let $\beta > 1$ be a Pisot number and let α be normal to base β. Then the sequence $(\{\beta^n \alpha\})_{n \geq 1}$ does not have non-trivial bounded remainder sets.

strongly non-bounded remainder sets

Definition

Let $(x_n)_{n\geq 1}$ be a sequence in $[0,1)$. An interval $[a,b) \subseteq [0,1)$ is said to be a *strongly non-bounded remainder set (S-NBRS)* if for all $K \in \mathbb{N}$ there are K successive elements of $(x_n)_{n\geq 1}$ which all are contained in $[a,b)$ or which all are not contained in $[a,b)$.

strongly non-bounded remainder sets

Definition

Let $(x_n)_{n\geq 1}$ be a sequence in $[0,1)$. An interval $[a,b) \subseteq [0,1)$ is said to be a *strongly non-bounded remainder set (S-NBRS)* if for all $K \in \mathbb{N}$ there are K successive elements of $(x_n)_{n\geq 1}$ which all are contained in $[a,b)$ or which all are not contained in $[a,b)$.

If $(x_n)_{n\geq 1}$ has a dense collection of BRS, then there do not exist any S-NBRS.

$a_n = n + q_n$

Consider $J := [\{k_1\alpha\}, \{k_2\alpha\})$, $\{k_1\alpha\} < \{k_2\alpha\}$ which is a BRS for $(\{n\alpha\})_{n\geq 1}$. There exists $k \in \mathbb{N}$ such that

$$\{a_n\alpha\} \in J \Leftrightarrow \{n\alpha\} \in J \qquad \text{for all } n \geq k,$$

where $a_n = q_n + n$, because

$a_n = n + q_n$

Consider $J := [\{k_1\alpha\}, \{k_2\alpha\})$, $\{k_1\alpha\} < \{k_2\alpha\}$ which is a BRS for $(\{n\alpha\})_{n\geq 1}$. There exists $k \in \mathbb{N}$ such that

$$\{a_n\alpha\} \in J \Leftrightarrow \{n\alpha\} \in J \qquad \text{for all } n \geq k,$$

where $a_n = q_n + n$, because

$$\{n\alpha\} - \{a_n\alpha\} \mod 1 = \|q_n\alpha\|,$$

$a_n = n + q_n$

Consider $J := [\{k_1\alpha\}, \{k_2\alpha\})$, $\{k_1\alpha\} < \{k_2\alpha\}$ which is a BRS for $(\{n\alpha\})_{n\geq 1}$. There exists $k \in \mathbb{N}$ such that

$$\{a_n\alpha\} \in J \Leftrightarrow \{n\alpha\} \in J \qquad \text{for all } n \geq k,$$

where $a_n = q_n + n$, because

$$\{n\alpha\} - \{a_n\alpha\} \mod 1 = \|q_n\alpha\|,$$

and if $n \geq k_1, k_2$,

$$\{n\alpha\} - \{k_i\alpha\} \mod 1 \geq \min_{1\leq l\leq n} \|l\alpha\|.$$

$a_n = n + q_n$

Consider $J := [\{k_1\alpha\}, \{k_2\alpha\})$, $\{k_1\alpha\} < \{k_2\alpha\}$ which is a BRS for $(\{n\alpha\})_{n\geq 1}$. There exists $k \in \mathbb{N}$ such that

$$\{a_n\alpha\} \in J \Leftrightarrow \{n\alpha\} \in J \qquad \text{for all } n \geq k,$$

where $a_n = q_n + n$, because

$$\{n\alpha\} - \{a_n\alpha\} \mod 1 = \|q_n\alpha\|,$$

and if $n \geq k_1, k_2$,

$$\{n\alpha\} - \{k_i\alpha\} \mod 1 \geq \min_{1\leq l\leq n} \|l\alpha\|.$$

$\Rightarrow$ J is a BRS for $(\{a_n\alpha\})_{n\geq 1}$

$a_n = n + q_n$

Theorem (K., Larcher)

1. *Let* α *be irrational and let* q_n *be the* n*-th convergent from the continued fraction expansion of* α*. Then every interval of the form* $[\{k_1\alpha\}, \{k_2\alpha\})$ *with* $k_1, k_2 \in \mathbb{N}$ *and* $\{k_1\alpha\} < \{k_2\alpha\}$ *is a BRS for the uniformly distributed sequence* $(\{(q_n + n)\alpha\})_{n\geq 1}$*.*

$\boldsymbol{a_n = n + q_n}$

Theorem (K., Larcher)

1. *Let* α *be irrational and let* q_n *be the* n*-th convergent from the continued fraction expansion of* α*. Then every interval of the form* $[\{k_1\alpha\}, \{k_2\alpha\})$ *with* $k_1, k_2 \in \mathbb{N}$ *and* $\{k_1\alpha\} < \{k_2\alpha\}$ *is a BRS for the uniformly distributed sequence* $(\{(q_n + n)\alpha\})_{n\geq 1}$*.*
2. *In general the sequence* $(\{(q_n + n)\alpha\})_{n\geq 1}$ *does not have the same collection of BRS as the sequence* $(\{n\alpha\})_{n\geq 1}$*.*

growth rates of a_n

growth rates of a_n

- a_n grows slower than linearly $\Rightarrow$ no BRS

Let $(a_n)_{n\geq 1}$ be an increasing sequence of integers and let $(\{a_n\alpha\})_{n\geq 1}$ be uniformly distributed. If

$$\lim_{n\to\infty} \frac{a_n}{n} = 0,$$

then every interval $[a,b) \subset [0,1)$ of positive measure is a S-NBRS.

growth rates of a_n

- every other growth rate $\Rightarrow$ dense collection of BRS

strictly increasing sequence $\varphi(n) \geq 2n$ for all $n \in \mathbb{N}$

α irrational with bounded continued fraction coefficients

growth rates of a_n

- every other growth rate $\Rightarrow$ dense collection of BRS

strictly increasing sequence $\varphi(n) \geq 2n$ for all $n \in \mathbb{N}$

α irrational with bounded continued fraction coefficients

$\Rightarrow$ there exists a strictly increasing sequence of integers $(a_n)_{n\geq 1}$, with

$$\varphi(n) \leq a_n \leq L\varphi(n)$$

for all $n \geq 1$ such that $(\{a_n\alpha\})_{n\geq 1}$ has a dense collection of BRS and is u. d.

an extension

$\mathcal{A}$ is a countable set of irrational numbers

$\Rightarrow$ there is a lacunary sequence $(a_n)_{n\geq 1}$ of integers,

$$a_n = n + q_n'$$

such that $(\{a_n\alpha\})_{n\geq 1}$ has a dense collection of BRS for all $\alpha \in \mathcal{A}$ and each of those sequences is u. d.

$a_n = f(n)$

- BRS for $(\{n^2\alpha\})_{n\geq 1}$?
- BRS for $(\{f(n)\alpha\})_{n\geq 1}$ with $f(x) \in \mathbb{Z}[x]$?

$a_n = f(n)$

- BRS for $(\{n^2\alpha\})_{n\geq 1}$?
- BRS for $(\{f(n)\alpha\})_{n\geq 1}$ with $f(x) \in \mathbb{Z}[x]$?

Theorem (Liardet, 1978)

Let $p(x) \in \mathbb{R}[x]$ with degree $d \geq 2$ and an irrational leading coefficient. Then the only BRS of the sequence $(\{p(n)\})_{n\geq 1}$ are the empty set and $[0, 1)$.

$a_n = f(n)$

- BRS for $(\{n^2\alpha\})_{n\geq 1}$?
- BRS for $(\{f(n)\alpha\})_{n\geq 1}$ with $f(x) \in \mathbb{Z}[x]$?

Theorem (Liardet, 1978)

Let $p(x) \in \mathbb{R}[x]$ with degree $d \geq 2$ and an irrational leading coefficient. Then the only BRS of the sequence $(\{p(n)\})_{n\geq 1}$ are the empty set and $[0, 1)$.

- elementary proof?

$a_n = f(n)$

- elementary proof?

Theorem (K., Larcher)

Let $\boldsymbol{p}(n) = (p^{(1)}(n), p^{(2)}(n), \ldots, p^{(s)}(n))$*, where all* $p^{(i)}(n)$*,* $i \leq s$ *are real polynomials with the property, that for each point* $\boldsymbol{h} \in \mathbb{Z}^s, \boldsymbol{h} \neq 0$*, the polynomial* $\langle \boldsymbol{h}, \boldsymbol{p}(n) \rangle$ *has at least one non-constant term with irrational coefficient. Then the sequence* $(\{\boldsymbol{p}(n)\})_{n \geq 1}$ *does not have any S-NBRS.*