Large values of short character sums

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1. Introduction

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are often very useful. The Polya-Vinogradov inequality states that sum over any interval is $O(\sqrt{q}\ln q)$ and Burgess's bound gives a nontrivial estimate $O(Np^{-c(\varepsilon)})$ provided $N \geq p^{1/4+\varepsilon}$.

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and some heuristics also show that q^{ε} can be replaced by

$$\exp\left(O\left(\sqrt{\frac{\ln q}{\ln \ln q}}\right)\right)$$

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$$\left(\frac{n}{p}\right) = \begin{cases} +1 \text{ if } n \text{ is quadratic residue mod } p \\ -1 \text{ if } n \text{ is nonresidue} \\ 0 \text{ if } n \text{ is divisible by } p \end{cases}$$

More generally, every primitive quadratic character can be written as $\left(\frac{D}{n}\right)$ where D is a fundamental discriminant.

It turns out that conditional bound that we mentioned before can be substantially improved:

Theorem 1 (Granville, Soundararajan, 2001)

Let $\omega(q)$ be any function that tends to infinity for $q \to +\infty$. If Generalized Riemann Hypothesis is true then

$$\sum_{n \le x} \chi(n) = o(x)$$

for all primitive characters to the modulus q and $x = (\ln q)^{\omega(q)}$.

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Theorem 2 (Granville, Soundararajan, 2001)

For all sufficiently large q and any $A > 0$ there exists a fundamental discriminant D with $q \le |D| \le 2q$ such that the inequality

$$\sum_{n \le x} \left(\frac{D}{n}\right) \gg x$$

holds, where $x = (\frac{1}{3} \ln q)^A$.

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Let $A \geq 1$ *be an arbitrarily large fixed number,* $x \geq x_0(A)$ *and* $y = (\ln x)^A$*. Then there exists a prime number* p *with* $x < p \leq 2x$ *such that*

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So, sums of Legendre characters of length $(\ln p)^A$ are not $o((\ln p)^A)$. Proof of this result uses some properties of Siegel zeros of Dirichlet L-functions.

4. Ideas of proof

Let us denote $S(p,y) = \sum_{n \le y} \left(\frac{n}{p}\right)$. One might try to compute the average value of $S(p,y)$ over primes and hope to get something large, but it turns out that it is much smaller than y (it is of order $\sqrt{y}$ by Siegel-Walfisz theorem).

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So, our main idea is to introduce some weights w_p that are very biased towards large values of $S(p,y)$.
My choice of w_p is as follows:

$$w_p = \prod_{q \le M} \left(1 + \left(\frac{q}{p}\right)\right) = \begin{cases} 2^{\pi(M)} \text{ if for all primes } q \le M \text{ we have } \left(\frac{q}{p}\right) = 1 \\ 0 \text{ everywhere else} \end{cases}$$

Here $M = (\ln x)^{1/3}$.

So, the main goal is to prove that

$$\frac{\sum\limits_{x<p\leq 2x} S(p,y)w_p \ln p}{\sum\limits_{x<p\leq 2x} w_p \ln p} \gg y$$

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Let us denote the numerator and denominator of this expression $S_0(x)$ and $S_1(x,A)$.
To provide an asymptotic formulas for S_0 and S_1 we need to deal with zeros of some Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{+\infty} \chi(n) n^{-s}$$

We will use the following two classical theorems

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Theorem 4

For some constants c and $c_1 > 0$ and any nonprincipal character χ to the modulus $q \le e^{c_1\sqrt{\ln x}}$ we have

$$\sum_{p \le x} \chi(p) \ln p = -\delta \frac{x^\beta}{\beta} + O\left(xe^{-c_1\sqrt{\ln x}}\right).$$

Where $\delta = 1$ if $L(s,\chi)$ has a real zero β with $\beta > 1 - \frac{c}{\ln q}$ and $\delta = 0$ otherwise.

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Where $\delta = 1$ if $L(s,\chi)$ has a real zero β with $\beta > 1 - \frac{c}{\ln q}$ and $\delta = 0$ otherwise.

These exceptional real zeros are called Siegel zeros.

Theorem 5

Let χ_1 *and* χ_2 *be two different real characters to the moduli* q_1 *and* q_2. *Let* β_1 *and* β_2 *be a real zeros of* $L(s,\chi_1)$ *and* $L(s,\chi_2)$ *repectively. Then we have*

$$\max\{\beta_1,\beta_2\} \leq 1 - \frac{c_2}{\sqrt{\max\{q_1,q_2\}}}$$

and

$$\min\{\beta_1,\beta_2\} \leq 1 - \frac{c_3}{\ln(q_1 q_2)}$$

So, Siegel zeros cannot be too large and they are quite rare.

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So, Siegel zeros cannot be too large and they are quite rare. In particular, one can show that a family of L-functions $L(s, \left(\frac{f}{\cdot}\right))$, where $f \le e^{2M}$ is squarefree can have at most one really large exceptional zero.

We denote the corresponding squarefree integer by f and the real zero by β_f.

Using Theorems 4 and 5 one can obtain the following asymptotic formulas:

$$S_0(x) \sim x - \delta_1 \frac{(2x)^{\beta_f} - x^{\beta_f}}{\beta_f}$$

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$$S_1(x, A) \sim x r_{A,x}(1) - \delta_2 \frac{(2x)^{\beta_f} - x^{\beta_f}}{\beta_f} r_{A,x}(f),$$

where $\delta_2 = 1$ if f exists and 0 if it is not. Here $r_{A,x}$ is defined by the formula

$$r_{A,x}(c) = \#\{(a, d) : 1 \leq a \leq y, d \mid P(M), ad = ct^2 \text{ for some integer } t\}$$

To prove Theorem 3 we need the following lemma about $r_{A,x}$

Lemma 6

Let c be squarefree. Then $r_{A,x}(c) \ll \frac{y \ln y}{M}$ if $c \nmid P(M)$ and $r_{A,x}(c) = r_{A,x}(1) \gg y$ otherwise.

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- f doesn't exist. Then $\delta_1 = \delta_2 = 0$ and we get

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therefore we are done.

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- f exists but it doesn't divide $P(M)$. Then $\delta_1 = 0$ but $\delta_2 = 1$ and we have

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Lemma 6 shows that $r_{A,x}(f)$ is much smaller than $r_{A,x}(1)$, therefore we get

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- f exists and divides $P(M)$. Then $\delta_1 = \delta_2 = 1$ and so we can get no nice asymptotic formula for S_0 and S_1. We only get the following facts:

$$S_0(x) \sim x - \frac{(2x)^{\beta_f} - x^{\beta_f}}{\beta_f}$$

  and

$$S_1(x, A) \sim x r_{A,x}(1) - \frac{(2x)^{\beta_f} - x^{\beta_f}}{\beta_f} r_{A,x}(f).$$

In this final case Lemma 6 gives us $r_{A,x}(1) = r_{A,x}(f)$ and so asymptotics for our sums are spoiled in exactly the same way!

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This concludes the proof.

Thank you for your attention!