Statistical properties of Klein polyhedra

Andrei Illarionov

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$$\Gamma_\alpha = \{(Q, \alpha Q - P) : \ P, Q \in \mathbb{Z}\};$$

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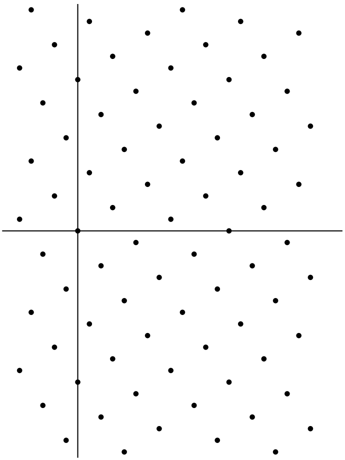

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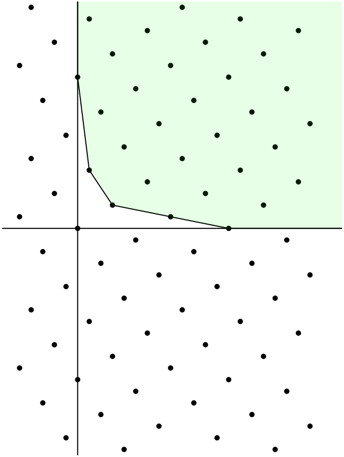

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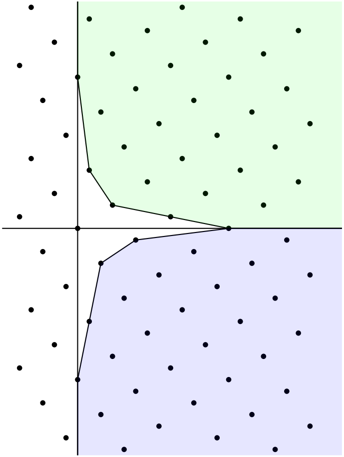

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$\alpha = [q_0; q_1, q_2, q_3, \ldots]$ (continued fraction decomposition);

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$$\pm(0,1), \qquad (Q_n, \alpha Q_n - P_n),\ n = 0, 1, 2, \ldots .$$

- If $[a, b]$ is a side (edge) of $K(\alpha)$, then there exists n such that

$$a = (Q_{n-1}, \alpha Q_{n-1} - P_{n-1}), \qquad b = (Q_{n+1}, \alpha Q_{n+1} - P_{n+1}),$$
$$q_{n+1} = \#(\Gamma \cap (a, b]) \quad \text{(integer lenth of segment } [a, b]). \qquad (1)$$

(#X is the number of elements in a finite set X).

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Definition (F. Klein 1895)

The polyhedron

$$K_\theta(\Gamma) = \text{Conv}\,\{\gamma \in \Gamma \setminus \{0\} : \quad \theta_i \gamma_i \geq 0, \quad i = \overline{1, s}\}.$$

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$$K(\Gamma) = \bigcup_\theta K_\theta(\Gamma).$$

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- A.A. Illarionov, Sbornik: Mathematics (2013, 2015, 2018)

  (Some statistical properties of Klein polyhedra are examined)

The average value of number of vertices

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The average value of the number of vertices of Klein polyhedra for s-dimensional integer lattices with determinant N is equal to

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From results of H. Heilbronn, J. Porter (average length of a continuous fraction), it follows that

$$\text{(case } s = 2) \qquad E_0(N;2) = \frac{4\ln 2}{\zeta(2)} \ln N + O(\ln\ln N). \tag{2}$$

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Theorem (A.I., 2013)

The asymptotical formula for average number of vertices of 3D Klein polyhedra holds

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$C_0(3)$ is a positive constant. It is defined by 6D integral.

$$C_0(3) \approx 0.41205 \quad (4)$$

The proof is based on the parametrization of the vertices by the basis matrices of the corresponding lattice.

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Conjecture

There exists a positive constant $C_0(s)$ depending only on s such that

$$E_0(N; s) \sim C_0(s) \cdot \ln^{s-1} N \quad \text{at } N \to +\infty. \qquad (6)$$

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Define the average number of such faces by the formula

$$E_{1,k}(N; s) = \frac{1}{\#L_s(N)} \sum_{\Gamma \in L_s(N)} f_{1,k}(\Gamma)$$

($L_s(N)$ is the set of s-dimensional integer lattices Γ with $\det \Gamma = N$)

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H. Heilbronn (1969) proved the following asymptotic formula for the frequency of occurrence of a given natural k as a partial quotient:

$$\frac{1}{\#\Phi(Q)} \sum_{P \in \Phi(Q)} \#\{n : q_n(P/Q) = k\} = C(k) \ln Q + O_k\left((\ln \ln Q)^4\right),$$

$$C(k) = \frac{2}{\zeta(2)} \cdot \ln\left(1 + \frac{1}{k(k-2)}\right), \quad \Phi(Q) = \{P \in [1, Q] : \ (P, Q) = 1\}.$$

Using this result we can prove that

$$(\text{case } s = 2) \qquad E_{1,k}(N; 2) = C_{1,k}(2) \cdot \ln N + O_k(\ln \ln N), \qquad (8)$$

$$C_{1,k}(2) = 2C(k) \asymp \frac{1}{k^2}. \qquad (9)$$

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Theorem (A.I., 2015)

For any $\underline{k > 1}$, we have the asymptotical formula for average number of edges of 3D Klein polyhedra

$$E_{1,k}(N;3) = C_{1,k}(3) \cdot \ln^2 N + O_k(\ln N \cdot \ln\ln N), \tag{10}$$

$$C_{1,k}(3) = \frac{6}{\zeta(2)\zeta(3)} \cdot \frac{1}{k^3} + O\left(\frac{1}{k^4}\right) \asymp \frac{1}{k^3}. \tag{11}$$

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Problem: $k = 1$?

If relation (10) holds for $k = 1$, then we have asymptotical formulas for average values of number of vertices, edges, facets (i.e. f-vector) of 3D Klein polyhedra.

A polyhedron (polygon) is said to be integer (integral) if all its vertices are integer points.

Integral polyhedra P_1, $P_2 \subset \mathbb{R}^s$ are said to be integer-linear equivalent (has the same integer-linear type) if there is a linear map L such that

$$L : \mathbb{R}^s \to \mathbb{R}^s, \quad L\mathbb{Z}^s = \mathbb{Z}^s, \quad LP_1 = P_2.$$

The integer-linear type of a polyhedron depends on the number and location of integer points contained in this polyhedron.

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Definition

Let T be an integer d-dimensional integral polyhedron.

We say that F is a T-type face, if LF is integer-linear equivalent to T.

Example

Let $d = 1$, i.e. F is a 1-dimensional face (edge). Then the type of F is unique defined by the value

$$k = \#(F \cap \Gamma) - 1.$$

If $\dim \Gamma = 2$, $\Gamma = \Gamma_\alpha = \{(Q, \alpha Q - P) : \ P, Q \in \mathbb{Z}\}$, then k is equal to corresponding q_{k+1} (partial quotient for α).

The type of a face depends on the number and location of lattices points contained in this face.

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Let $f(\Gamma; T)$ be the number of T-type faces of Klein polyhedra $K(\Gamma)$. Define the average number of such faces

$$E_T(N, s) = \frac{1}{\#L_s(N)} \sum_{\Gamma \in L_s(N)} f(\Gamma, T).$$

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Let $s = 2$. Then the type T of facet is unique defined by the number of lattice points laying at the face. Let $k = \#(T \cap \mathbb{Z}^2) - 1$. Then

$$\text{(case } s = 2) \qquad E_T(N; 2) = E_{1,k}(N; 2) \sim 2 \cdot C(k) \cdot \ln N, \quad N \to \infty. \qquad (12)$$

The average value of number of T-type facets. Case $s = 3$

Let TF_s be the set of $(s-1)$-dimensional integral polyhedra $T \subset \mathbb{R}^s$ such that there is a s-dimensional Klein polyhedron having T-type facet.

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Theorem (A.I., 2013)

The asymptotical formula for average value of number of T-type facets of 3D Klein polyhedra holds

$$\forall T \in TF_3 \qquad E_T(N;3) = C_T(3) \cdot \ln^2 N + O_T(\ln N \cdot \ln\ln N). \qquad (13)$$

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Then

$$E_T(N; s) = C_T(s) \cdot \ln^{s-1} N + O_T(\ln^{s-2} N \cdot \ln \ln N). \qquad (14)$$

The constant $C_T(s)$ is defined by $(s^2 - s)$-dimensional integral

The constant $C_T(s)$

Let $\mathrm{D}_s(\mathbb{R}) \subset \mathrm{GL}_s(\mathbb{R})$ be the set of diagonal $(s \times s)$-matrices.
Let $\mathrm{P}_s(\mathbb{R}) = \mathrm{D}_s(\mathbb{R}_+)\backslash\mathrm{GL}_s(\mathbb{R})$.
Let $\mathcal{P}$ be the projectivisation mapping $\mathrm{GL}_s(\mathbb{R}) \to \mathrm{P}_s(\mathbb{R})$.

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Let $\mathcal{P}$ be the projectivisation mapping $\mathrm{GL}_s(\mathbb{R}) \to \mathrm{P}_s(\mathbb{R})$.
Define the measure μ on the $\mathrm{P}_s(\mathbb{R})$.
Let $\mathrm{GL}_s(\mathbb{R}, k) = \{X \in \mathrm{GL}_s(\mathbb{R}) : x_{ik_i} = 1,\ i = \overline{1, s}\}$.
The set of all $\mathrm{GL}_s(\mathbb{R}, k)$ forms an atlas of the manifold $\mathrm{P}_s(\mathbb{R})$, and matrices from $\mathrm{GL}_s(\mathbb{R}, k)$ are the coordinates of the corresponding elements of $\mathrm{P}_s(\mathbb{R}, k)$.
If $W \subset \mathrm{GL}_s(\mathbb{R}, k)$, and $w = \mathcal{P}(W)$, then

$$\mu(w) = \int_W \frac{dX}{|\det X|^s}.$$

The constant $C_T(s)$

For any $T \in TF_s$ there is a matrix set $\Omega_T \subset \mathrm{GL}_s(\mathbb{R})$ such that

$$\mathrm{D}_s(\mathbb{R}) \cdot \Omega_T = \Omega_T,$$

$$C_T(s) = \frac{\mu(\mathcal{P}(\Omega_T))}{\zeta(2)\zeta(3)\dots\zeta(s)} \frac{1}{(s-1)!}.$$

The condition A)

A) the relative interior of the polyhedron T has at least one integer point.

Let us consider the polyhedron T as a n-dimensional integral polyhedron from $\mathbb{R}^n$ ($n = s - 1$).
Let the set $U(T)$ consists of $x \in \mathbb{R}^n$ such that

$$\mathrm{Conv}\,(T \cup \{x\}) \cap \mathbb{Z}^n = (T \cup \{x\}) \cap \mathbb{Z}^n.$$

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Question:

Is finite the set $U(T) \cap \mathbb{Z}^n$?

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(case $n = 2$ ($s = 3$))

Let Δ be a $2D$ integral empty symplex. Then $\operatorname{mes} \Delta = 1/2$.

From this property, it follows that

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(case $n \geq 3$ ($s \geq 4$))

Let $n \geq 3$. For any $R > 1$ there is a n-dimensional integral empty symplex Δ such that $\operatorname{mes} \Delta > R$.

From this property, it follows that

$$U(T) \cap \mathbb{Z}^n \text{ may be infinite (if } n \geq 3, \text{ i.e. } s \geq 4).$$

Lemma

Let T be a n-dimensional integral polyhedron from $\mathbb{R}^n$.

Suppose that interior of T contents at least one integer point.

Then the set $U(T)$ is bounded ($\Longrightarrow U(T) \cap \mathbb{Z}^n$ is finite).

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    1. What is the typical vertex degree?
    2. How many sides does a typical face have?
    3. What is the integral area of a typical face? of Klein polyhedron boundary?
2. The almost all presented results were obtained for 3D Klein polyhedra. We know almost nothing about s-dimensional Klein polyhedra at $s > 3$.

Thank you for your attention!