Asymptotic behaviour of the Sudler product of sines and a conjecture of Lubinsky

Sigrid Grepstad

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Joint work with Mario Neumüller and Lisa Kaltenböck

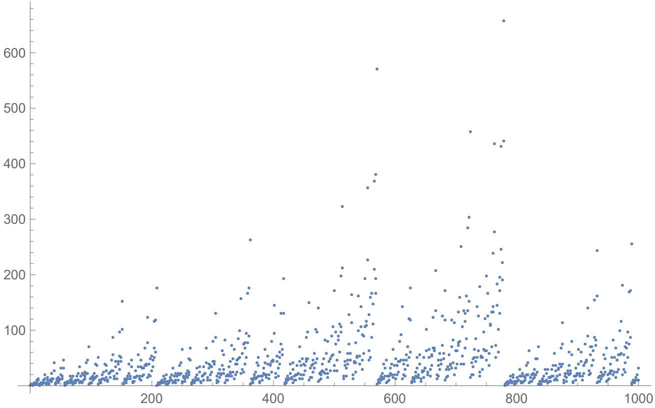
(Sudler's) Sine Product

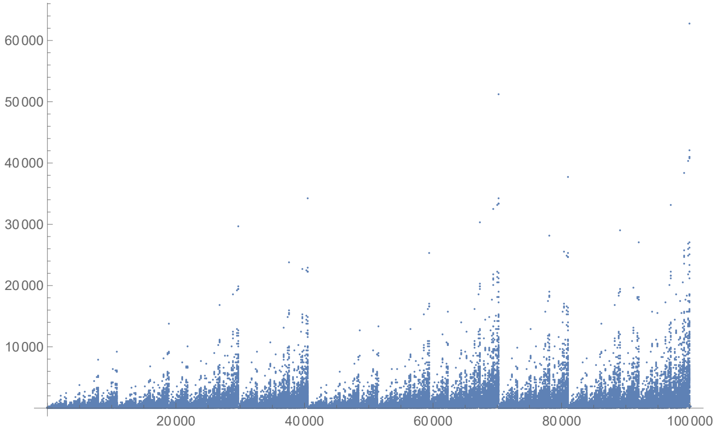
Let $\alpha \in \mathbb{R}$, and denote by $P_n(\alpha)$ the sine product

$$P_n(\alpha) = \prod_{r=1}^{n} |2 \sin \pi r \alpha| \,.$$

Q: How does $P_n(\alpha)$ grow as $n \to \infty$?

Suppose we let $\alpha = \sqrt{3}$.





Q: Is $P_n(\sqrt{3}) = \mathcal{O}(n)$ as $n \to \infty$?

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- For all irrational α, we have

$$\limsup_{n\to\infty} \frac{\log P_n(\alpha)}{\log n} \geq 1.$$

Dependence on continued fraction coefficients

Let

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

Theorem (Lubinsky)

If $\sup_j a_j < \infty$, then there exist constants $C_1, C_2 > 0$ such that

$$n^{-C_2} \leq P_n(\alpha) \leq n^{C_1}.$$

Dependence on continued fraction coefficients

Theorem (Lubinsky)

If $\sup_j a_j = \infty$, then

$$\liminf_{n\to\infty} \log P_n(\alpha) = -\infty. \tag{1}$$

In fact, $P_n(\alpha)$ will decay to zero for infinitely many n faster than any power of n.

Quote Lubinsky (1999): "...and we are certain that (1) is true in general."

The special case $\alpha = (\sqrt{5}-1)/2$

Let $\alpha = \varphi = (\sqrt{5}-1)/2 = [0; 1, 1, 1, \ldots] = [0; \overline{1}]$.

Denote by $F_n = (1, 1, 2, 3, 5, 8, \ \ldots)$ the Fibonacci sequence.

Consider the subsequence

$$P_{F_n}(\varphi) = \prod_{r=1}^{F_n} |2 \sin \pi r \varphi| \, .$$

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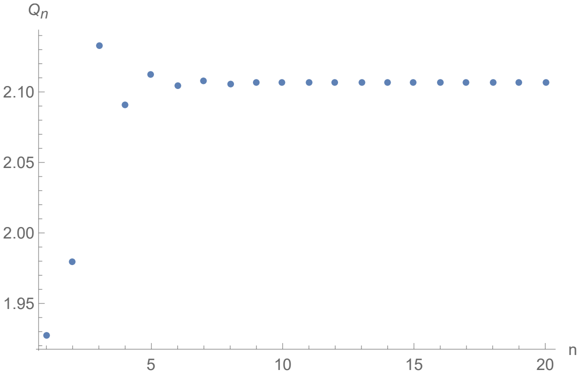
Theorem (Mestel and Verschueren 2016)

$$\lim_{n \to \infty} P_{F_n}(\varphi) = c > 0$$

Periodic continued fraction expansions

Let $\alpha = \{\sqrt{2}\} = [0; \overline{2}] = [0; 2, 2, 2, \ldots]$.



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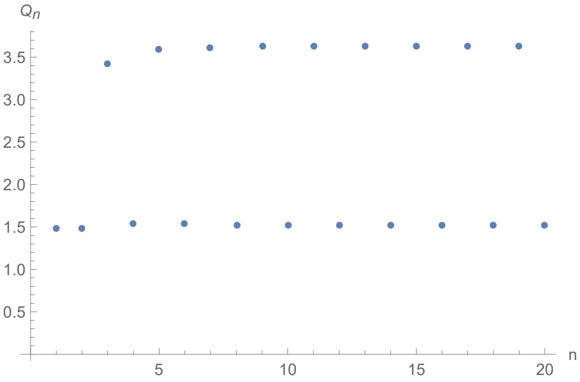
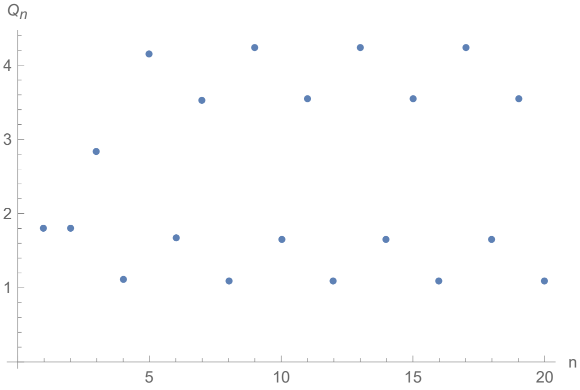


Periodic continued fraction expansions

Let $\alpha = \{\sqrt{7}\} = [0; \overline{1, 1, 1, 4}]$.



Periodic continued fraction expansions

Theorem (G. and Neumüller 2018)

Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$, and denote by q_n the nth best approximation denominator of α. Then

$$\lim_{m \to \infty} P_{q_{\ell m+k}}(\alpha) = \lim_{m \to \infty} \prod_{r=1}^{q_{\ell m+k}} |2 \sin \pi r \alpha| = c_k$$

for positive constants $c_1, \ldots, c_\ell$.

Corollary

Adding a preperiod to the continued fraction expansion of α does not change the conclusion in the theorem above.

Recovering Lubinsky's polynomial bounds for $\varphi = \frac{\sqrt{5}-1}{2}$

Let $n = \sum_{j=1}^{m} F_{n_j}$ be the Zeckendorf representation of n. Then we may rewrite $P_n(\varphi)$ as

$$P_n(\varphi) = \prod_{j=1}^{m} \prod_{r=1}^{F_{n_j}} |2 \sin \pi(r\varphi + k_j\varphi)| ,$$

with $k_j = \sum_{s=j+1}^{m} F_{n_s}$ for $1 \leq j \leq m-1$ and $k_m = 0$.

Note that $m = \mathcal{O}(\log n)$.

Recovering Lubinsky's polynomial bounds for $\varphi = \frac{\sqrt{5}-1}{2}$

All "inner products" can be bounded by real constants $0 < K_1 < 1 < K_2$:

$$K_1 \leq \prod_{r=1}^{F_{n_j}} |2 \sin \pi (r\varphi + k_j\varphi)| \leq K_2$$

Thus, we have

$$K_1^m \leq P_n(\varphi) \leq K_2^m.$$

Our strategy

Show that for sufficiently large values of n_j (or equivalently j), the inner product can be bounded below by

$$\prod_{r=1}^{F_{n_j}} |2 \sin \pi(r\varphi + k_j\varphi)| \geq 1, \quad j \geq J.$$

Accordingly, we have

$$P_n(\varphi) \geq K_1^J \quad \text{for all } n = 1, 2, \ldots$$

Theorem (G., Kaltenböck and Neumüller)

For $\varphi = (\sqrt{5} - 1)/2$, we have

$$P_n(\varphi) > c > 0 \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

In other words, it is *not* the case that

$$\liminf_{n \to \infty} P_n(\alpha) = 0$$

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Q: Does (2) hold for other quadratic irrationals?

Thank you for your attention.