Rényi $\alpha-$dimension of random variables with generalized Cantor distribution and Hausdorff dimension of generalized Cantor sets

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- A procedure to construct a generalized Cantor set and Cantor functions

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- Rényi $\alpha-$lower dimension of variables with generalized Cantor distribution

A procedure to construct a generalized Cantor set and Cantor function

$[a, b)$ interval in $\mathbb{R}$ and $q \geq 2$

The operation

$$\mathrm{Del}_q\big([a, b)\big) := [a, b) \setminus \Big[a + \frac{b-a}{q}, b - \frac{b-a}{q}\Big).$$

converts the interval $[a, b)$ into a pair of disjoint intervals.

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Start with $[0, 1]$ and iterate Del_q an infinite number of steps; at the $n-$th step apply Del_q to each one of the intervals obtained in the $(n-1)-$th step.

Definition

The $q-$Cantor set is the complement in $[0, 1]$ of the union of all the deleted intervals.

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Definition

The $q-$Cantor set is the complement in $[0, 1]$ of the union of all the deleted intervals.

Of course Del_3 produces the classical Cantor set.

Rules for numbering the deleted intervals

(a) Number the intervals deleted at step n with a string of symbols $\epsilon_{n-1}, \ldots, \epsilon_1, \epsilon_0$ (of length n), $\epsilon_j \in \{0, 1\}$, $\epsilon_0 = 1$.

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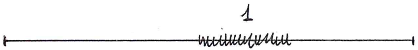
(b) If at the step n we have deleted an interval $I^{(q)}_{\epsilon_{n-1},\dots,\epsilon_1,1}$, the interval deleted at step $n+1$ on the left (resp. on the right) of $I^{(q)}_{\epsilon_{n-1},\dots,\epsilon_1,1}$ is denoted $I^{(q)}_{\epsilon_{n-1},\dots,\epsilon_1,0,1}$ (resp. $I^{(q)}_{\epsilon_{n-1},\dots,\epsilon_1,1,1}$).

First steps

- **Step 1**. Delete I_1.

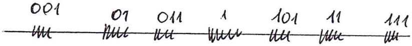


First steps

- **Step 2**. Delete I_{01} on the left of I_1 and I_{11} on the right of

First steps

- **Step 3**. Delete I_{001} on the left of I_{01} and I_{011} on the right of I_{01}; delete I_{101} on the left of I_{11} and I_{111} on the right of I_{11}.



The $q-$Cantor distribution

Definition

If $I_{\epsilon_{n-1},\dots,\epsilon_1,1}$ is a deleted interval,

$$x = \frac{1}{2^n}\left(1 + 2\epsilon_1 + 2^2\epsilon_2 + \cdots + 2^{n-1}\epsilon_{n-1}\right)$$

is the dyadic rational *associated* to $I_{\epsilon_{n-1},\dots,\epsilon_1,1}$.

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Definition

$\mathcal{C}_q$ is the continuous function that takes the value x on $I^{(q)}_{\epsilon_{n-1},\dots,\epsilon_1,1}$.

The $q-$Cantor measure

The function $\mathcal{C}_q$, being non–decreasing and continuous, is the distribution function of a continuous measure μ_q on $[0,1)$, that can be constructed in the following way. Let

$$\left\{ I^{(q)}_{\epsilon_{n-1},\dots,\epsilon_1,1} \right\}_{(\epsilon_{n-1},\dots,\epsilon_1,1)\in\mathcal{E}_n}$$

be the collection of intervals deleted at step n. Then the set

$$[0,1) \setminus \Big(\bigcup_{\substack{(\epsilon_{r-1},\dots,\epsilon_1,1)\in\mathcal{E}_r \\ r\le n}} I^{(q)}_{\epsilon_{r-1},\dots,\epsilon_1,1} \Big)$$

is the union of a disjoint family of intervals to each one of which μ_q attributes measure $\frac{1}{2^n}$.

In the paper [2] Rényi constructs a measure on $[0,1)$ in this way, but in his discussion the number q changes at each step of the deletion procedure.

The $\hat{q}-$Cantor measure

Following Rényi, allow the number q change at each step, i.e. take $\hat{\mathbf{q}} := (q_n)_{n\geq 1}$ (a sequence of numbers, $q_n \geq 2$); at step n apply Del_{q_n} to each one of the intervals obtained at step $n-1$. We obtain the $\hat{\mathbf{q}}-$*Cantor set*

$$\mathfrak{S}_{\hat{\mathbf{q}}} := [0,1) \setminus \Big(\bigcup_{n=1}^{\infty} \bigcup_{(\epsilon_{n-1},\dots,\epsilon_1,1)\in\mathcal{E}_n} I^{(q_n)}_{\epsilon_{n-1},\dots,\epsilon_1,1} \Big).$$

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Definition

The $\hat{\mathbf{q}}-$Cantor function $\mathcal{C}_{\hat{\mathbf{q}}}$ is the continuous function that takes the value x on $I^{(q_r)}_{\epsilon_{r-1},\dots,\epsilon_1,1}$, where x is the associated dyadic rational. $\mathcal{C}_{\hat{\mathbf{q}}}$ is the distribution function of a continuous measure $\mu_{\hat{\mathbf{q}}}$ on $[0,1)$, that attributes measure $\frac{1}{2^n}$ to each one of the intervals not deleted up to step n.

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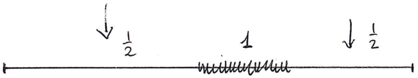

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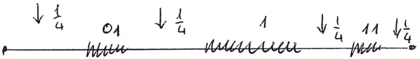

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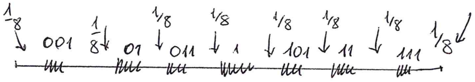

Endpoints of a deleted interval: a numerical example

I_{1011} is deleted at the 4th step.

Question

How can I write the endpoints of I_{1011} in terms of q_1, q_2, q_3, q_4? The associated dyadic rational is

$$x = \frac{1}{2^4}\left(1 + 2 \cdot 1 + 2^2 \cdot 0 + \cdots + 2^3 \cdot 1\right) = \frac{11}{2^4}.$$

There are exactly two ways of writing 11 as a sum of powers of 2 with alternating signs:

$$11 = 2^4 - 2^3 + 2^2 - 2^1 + 2^0 \Rightarrow \frac{11}{2^4} = \frac{1}{2^0} - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4};$$

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Procedure

Replace

$$\frac{1}{2^{k}} = \prod_{j=1}^{k} \frac{1}{2} \longrightarrow p_{k} = \prod_{j=1}^{k} \frac{1}{q_j}$$

and obtain

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Then

$$I_{1011} = (a, b].$$

Rényi 1−dimension of random variables

X real random variable.

$$X_n = \frac{1}{n}[nX],$$

$$p_{k,n} = P\Big(X_n = \frac{k}{n}\Big) = P\Big(\frac{k}{n} \leq X < \frac{k+1}{n}\Big), k \in \mathbb{Z}, n \in \mathbb{N}^*.$$

$$H_0^{(1)}(X_n) = -\sum_k p_{k,n} \log p_{k,n}$$

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$$\underline{d}^{(1)}(X) := \liminf_{n\to\infty} \frac{H_0^{(1)}(X_n)}{\log n}; \qquad \overline{d}^{(1)}(X) := \limsup_{n\to\infty} \frac{H_0^{(1)}(X_n)}{\log n};$$

$\underline{d}(X)$ (resp. $\overline{d}(X)$) is the 1-*lower dimension* (resp. 1-*upper dimension*) of X.

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Rényi has generalized the Shannon entropy to a family of entropies $H_0^{(\alpha)}$, $(\alpha \ge 0, \ne 1)$:

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Further, if $d^{(\alpha)}(X)$ exists,

$$H^{(\alpha)}_{d^{(\alpha)}(X)}(X) = \lim_{t\to+\infty} \left(H_0^{(\alpha)}(X_t) - d^{(\alpha)}(X)\log t\right),$$

provided the limit on the right exists.

$\hat{\mathbf{q}} = (q_n)_{n\geq 1}$ ($q_n \geq 2$ for all $\forall\, n$); $\mathcal{C}_{\hat{\mathbf{q}}}$ the associated $\hat{\mathbf{q}}-$ Cantor distribution function.

Notation

$$a_n = \log_2 q_n, \qquad s_n = \sum_{k=1}^{n} a_n;$$

hence

$$p_k = \prod_{j=1}^{k} \frac{1}{q_j} = \prod_{j=1}^{k} \frac{1}{2^{a_j}} = \frac{1}{2^{\sum_{j=1}^{k} a_j}} = \frac{1}{2^{s_k}}, \qquad k = 0, 1, 2, \ldots, n.$$

Holderianity of $\hat{\mathbf{q}}-$ Cantor distributions

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$$E := \left\{\lambda \in \mathbb{R} : (s_n - \lambda n)_{n\geq 1} \text{ is bounded from above}\right\}$$

is a (possibly empty) right half-line.

There are two crucial quantities:

$$\inf E = \limsup_{n\to\infty} \frac{s_n}{n} =: \ell \longleftarrow$$

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If $\sigma \in \mathbb{R}$, then ℓ is finite !

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Proposition

Let $\hat{\mathbf{q}} := (q_n)_{n\geq 1}$ be a given sequence of real numbers, with $q_n \geq 2$ for every n, and $\mathcal{C}_{\hat{\mathbf{q}}}$ the associated $\hat{\mathbf{q}}-$ Cantor function. Assume that $\sigma \in \mathbb{R}$ (hence ℓ as well!). Then

$$\mathcal{C}_{\hat{\mathbf{q}}}(t)^{\ell} \leq \max\{1, 2^{\sigma}\} t.$$

Examples

(i) $q_n = q \geq 2$, i.e. $a_n = \log_2 q := a \geq 1\ \forall\, n$. Here $s_n = an$, $\ell = a$, $\sigma = 0$ so $C(t)^{\ell} \leq t$ (this is proved in Lemma 2 of [1]).

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$$a_n = \frac{3 + (-1)^n}{2} = \begin{cases} 1 & \text{for odd } n \\ 2 & \text{for even } n. \end{cases}$$

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Here $s_n = \left[\frac{3}{2} n\right]$, $\ell = \frac{3}{2}$ and $\sigma = 0$. Again $C(t)^{\ell} \leq t$.
(iii) More generally, one can prove that the condition $\sigma \in \mathbb{R}$ is valid for any periodic sequence $(a_n)_{n \geq 1}$ (but not necessarily $\sigma = 0$!)

Holderianity of $\hat{\mathbf{q}}-$ Cantor distributions

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Corollary

Let $q_n \geq 3$ for every n. Then $\mathcal{C}_{\hat{\mathbf{q}}}$ is the first modulus of continuity of itself, i.e

$$\sup_{\substack{|x-y|\leq\delta \\ x,y\in[0,1]}} |\mathcal{C}_{\hat{\mathbf{q}}}(y) - \mathcal{C}_{\hat{\mathbf{q}}}(x)| = \mathcal{C}_{\hat{\mathbf{q}}}(\delta),$$

for all $\delta \in [0,1]$.

This is because a function f is a modulus of continuity if and only if it is defined, continuous, nondecreasing and subadditive on $[0,1]$ and $f(0) = 0$.

Conclusion

Theorem

Let $q_n \geq 3$ for every n and $\sigma < \infty$. Then $\mathcal{C}_{\hat{\mathbf{q}}}$ is holderian with exponent $\frac{1}{\ell}$, i.e.

$$\big|\mathcal{C}_{\hat{\mathbf{q}}}(y) - \mathcal{C}_{\hat{\mathbf{q}}}(x)\big| \leq \big(\max\{1, 2^{\sigma}\}\big)^{\frac{1}{\ell}} \big|x - y\big|^{\frac{1}{\ell}}.$$

The case that $q_n = 2$ for some n remains open! ☹

Remind

$$a_n = \log_2 q_n, \quad s_n = \sum_{k=1}^{n} a_n, \quad \ell = \limsup_{n\to\infty} \frac{s_n}{n}.$$

Hausdorff dimension of generalized Cantor sets

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Notation

$$S = \{s_1 < s_2 < \dots\}$$

$$\underline{d}(S) = \frac{1}{\ell} (\text{with } \underline{d}(S) = 0 \text{ if } \ell = +\infty)$$

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Proposition

The Hausdorff dimension of $\mathfrak{S}_{\hat{\mathbf{q}}}$ *(denoted* $d_H(\mathfrak{S}_{\hat{\mathbf{q}}})$*) is equal to* $\underline{d}(S)$.

Rényi $\alpha-$ lower dimension of variables with generalized Cantor distribution

Theorem

Let X be a random variable with the distribution $\mathcal{C}_{\hat{\mathbf{q}}}$. If $q_n \geq 3$ for every n and $\sigma \in \mathbb{R}$, then the $\alpha-$lower dimension of X equals $\underline{d}(S)$.

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To grasp the use of holderianity, i.e. $|\mathcal{C}(x) - \mathcal{C}(y)| \leq M|x-y|^{\underline{d}(S)}$ here is the proof of the inequality

$$\underline{d}^{(1)}(X) \geq \underline{d}(S)$$

i.e. case $\alpha = 1$ only.

$$H_0^{(1)}(X_t) = -\sum_k p_{k,t} \log p_{k,t}$$

$$= -\sum_k \left\{\mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right)\right\} \log \left\{\mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right)\right\}$$

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$$= -\sum_k \left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\} \log \left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\}$$

$$- \underline{d}(S) \sum_k \left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\} \log \frac{1}{t}$$

$$+ \underline{d}(S) \sum_k \left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\} \log \frac{1}{t}$$

$$= -\sum_k \left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\} \log \frac{\left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\}}{\left(\frac{1}{t}\right)^{\underline{d}(S)}} + \underline{d}(S) \log t$$

$$\geq -\sum_k \left\{ \mathcal{C}\left(\frac{k+1}{t}\right) - \mathcal{C}\left(\frac{k}{t}\right) \right\} \log M + \underline{d}(S) \log t$$

$$= -\log M + \underline{d}(S) \log t, \qquad \textit{by holderianity}.$$

We get

$$\underline{d}^{(1)}(X) = \liminf_{t \to +\infty} \frac{H_0^{(1)}(X_t)}{\log t} \geq \underline{d}(S).$$

Conclusion

Rényi $\alpha-$ lower dimension of variable with generalized Cantor distribution $=$

$=$ Hausdorff dimension of generalized Cantor set $=$

$$= \underline{d}(S), \qquad \forall \alpha \geq 0.$$

Some references

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☺ Thank you for attention! ☺