Multidimensional continued fractions and Diophantine exponents of lattices

UDT 2018, CIRM, Luminy

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Irrationality exponent vs Growth of partial quotients

Irrationality exponent $\theta \in \mathbb{R}\backslash\mathbb{Q}$

$$\mu(\theta) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; |\theta - p/q| \leqslant |q|^{-\gamma} \text{ admits } \infty \text{ solutions in } (q,p) \in \mathbb{Z}^2\right\}$$

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Relation to the growth of partial quotients

If

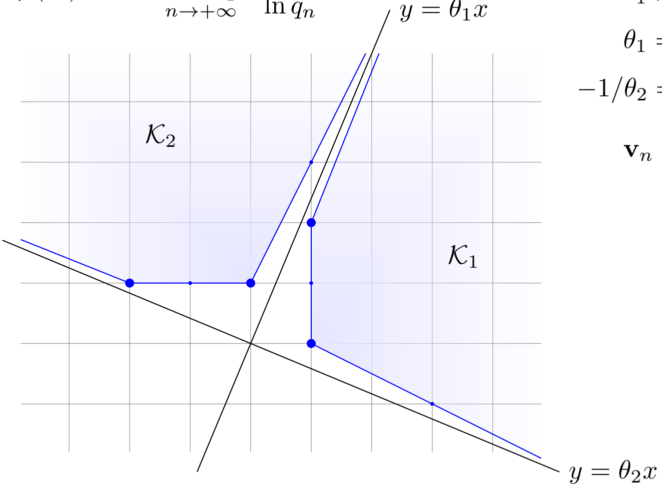
$$\theta = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}, \qquad \frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n],$$

then

$$\mu(\theta) - 2 = \limsup_{n \to +\infty} \frac{\ln a_{n+1}}{\ln q_n}$$

Geometry of continued fractions

$$\mu(\theta_1) - 2 = \limsup_{n\to+\infty} \frac{\ln a_{n+1}}{\ln q_n}$$



$$\theta_1 > 1, \quad -1 < \theta_2 < 0$$

$$\theta_1 = [a_0; a_1, a_2, \ldots]$$

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$$\mathbf{v}_n = \begin{pmatrix} q_n \\ p_n \end{pmatrix}, \quad \mathbf{v}_n = a_n \mathbf{v}_{n-1} + \mathbf{v}_{n-2}$$

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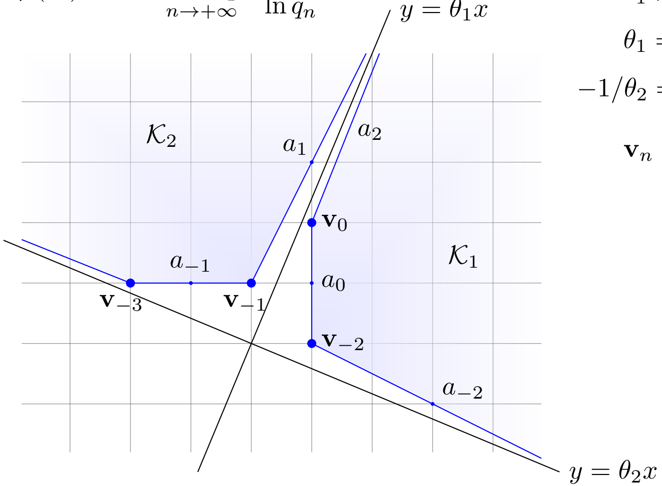

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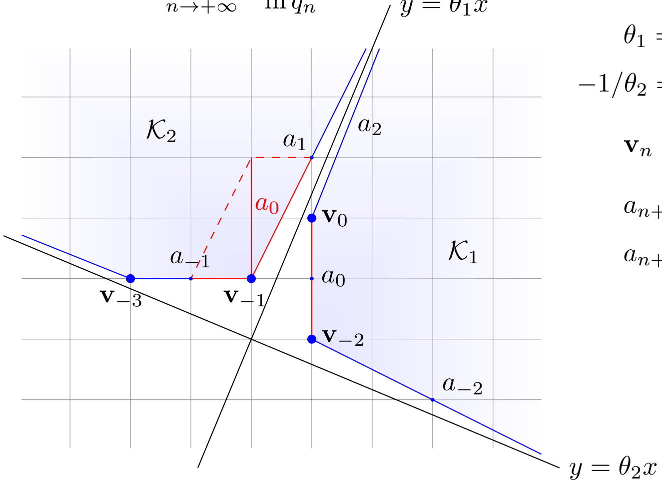

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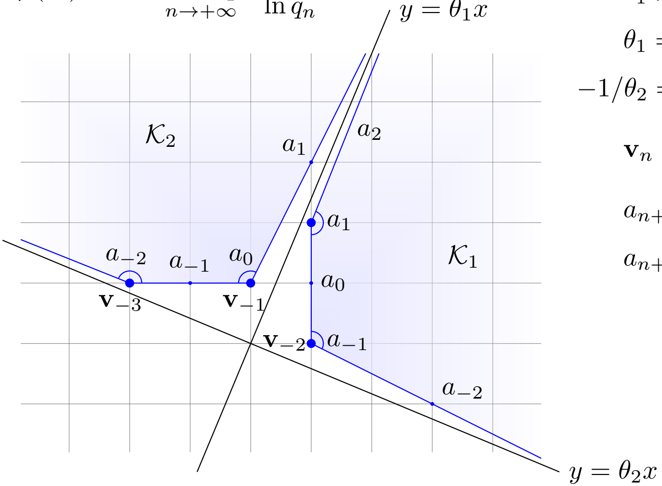

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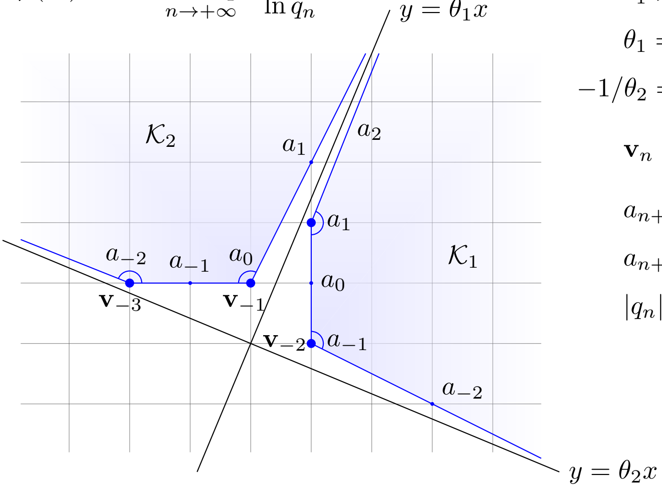

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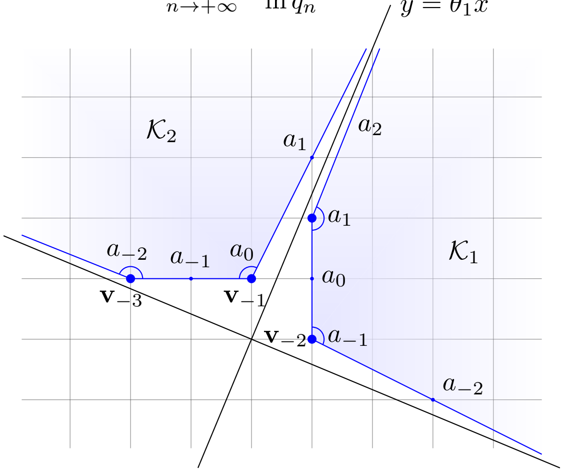

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$$L_1(q,p) = q\theta_1 - p, \quad L_2(q,p) = q\theta_2 - p$$

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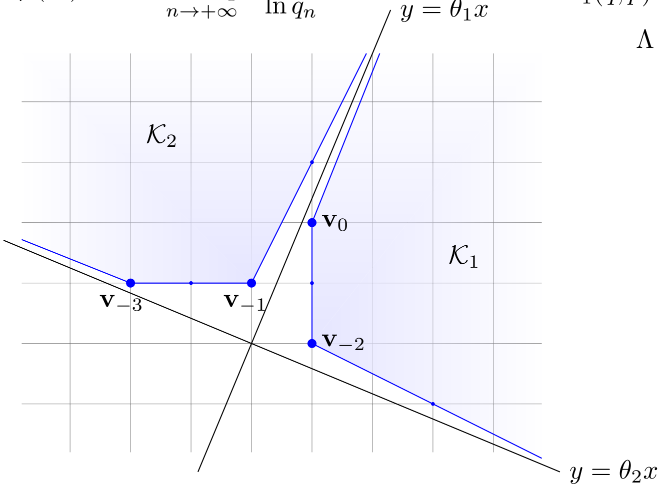

Irrationality exponent in terms of lattice exponents

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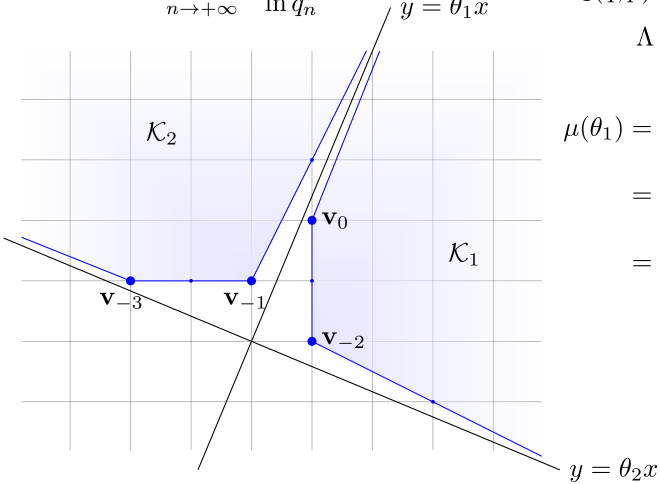

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$$= 2 + \limsup_{\substack{q,p\in\mathbb{Z} \\ |q|\to\infty}} \frac{\log\big(|q(q\theta_1 - p)|^{-1}\big)}{\log|q|} = $$

$$= 2 + \limsup_{n\to+\infty} \frac{\log\big(|L_1(\mathbf{v}_n)\cdot L_2(\mathbf{v}_n)|^{-1}\big)}{\log|\mathbf{v}_n|}$$

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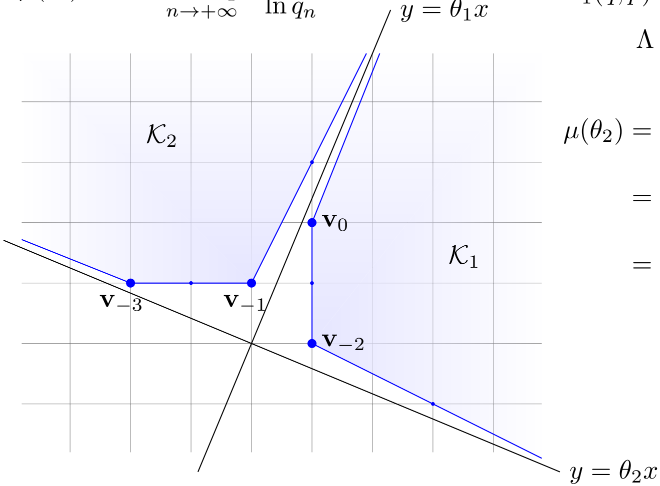

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$$\mu(\theta_2) = \limsup_{\substack{q,p\in\mathbb{Z} \\ |q|\to\infty}} \frac{\log\left(|\theta_2 - p/q|^{-1}\right)}{\log|q|} =$$

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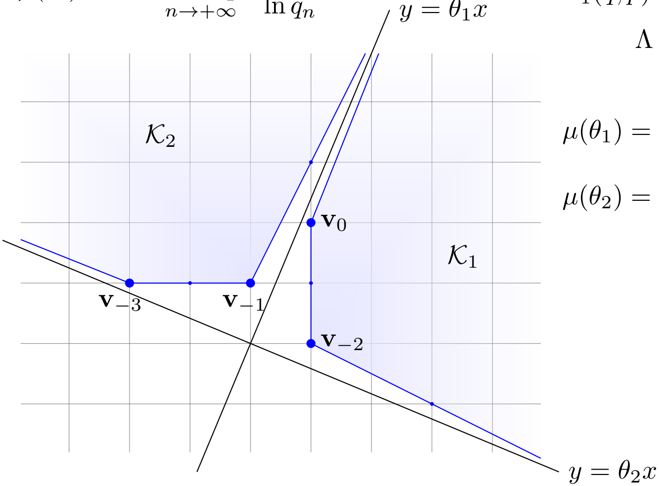

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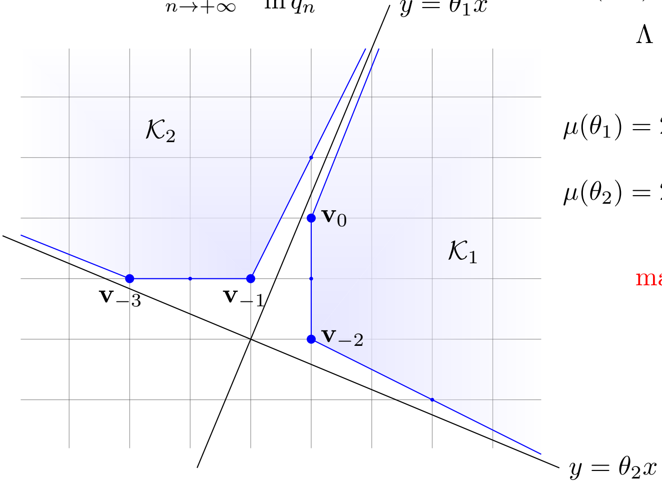

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$$\max(\mu(\theta_1), \mu(\theta_2)) = 2 + 2\omega(\Lambda)$$

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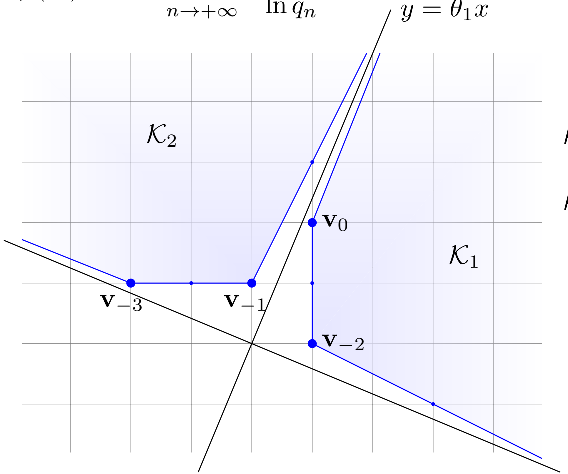

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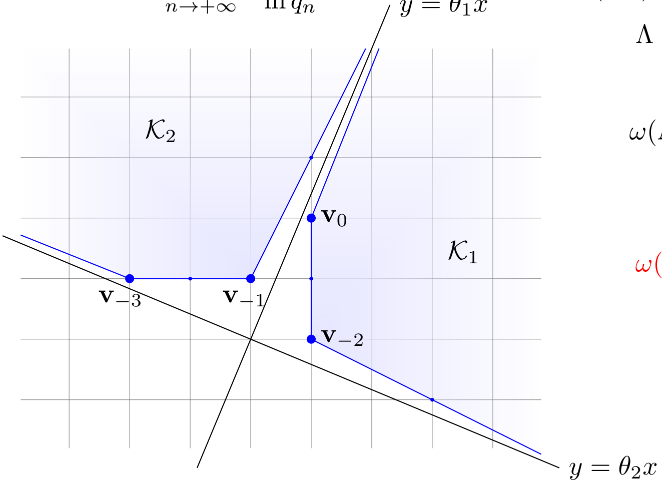

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$$\omega(\Lambda) = \frac{1}{2} \limsup_{\substack{|\mathbf{v}|\to\infty \\ \mathbf{v}\in\mathcal{V}(\mathcal{K}_1)\cup\mathcal{V}(\mathcal{K}_2)}} \frac{\log(\alpha(\mathbf{v}))}{\log|\mathbf{v}|}$$

Higher dimensions: lattice exponents and Klein polyhedra

Λ a lattice in $\mathbb{R}^d$, $\det \Lambda = 1$

$\Pi(\mathbf{x}) = |x_1 \dots x_d|^{1/d}$ for $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

Lattice exponent

$$\omega(\Lambda) = \limsup_{\substack{\mathbf{x}\in\Lambda \\ |\mathbf{x}|\to\infty}} \frac{\log\left(\Pi(\mathbf{x})^{-1}\right)}{\log|\mathbf{x}|} = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; \Pi(\mathbf{x}) \leqslant |\mathbf{x}|^{-\gamma} \text{ for infinitely many } \mathbf{x} \in \Lambda\right\}$$

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$\mathcal{O}$ an orthant

Klein polyhedron

$$\mathcal{K} = \operatorname{conv}\left(\mathcal{O}\cap\Lambda\backslash\{\mathbf{0}\}\right)$$

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Determinant of a face F

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the vertices of F. Then

$$\det F = \sum_{1\leqslant i_1<\ldots<i_d\leqslant k} \big|\det(\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_d})\big|$$

Determinant of an edge star $\mathrm{St}_{\mathbf{v}}$

Let $\mathbf{r}_1, \ldots, \mathbf{r}_k$ be the primitive vectors parallel to the edges incident to $\mathbf{v}$. Then

$$\det \mathrm{St}_{\mathbf{v}} = \sum_{1\leqslant i_1<\ldots<i_d\leqslant k} \big|\det(\mathbf{r}_{i_1}, \ldots, \mathbf{r}_{i_d})\big|$$

Λ a lattice in $\mathbb{R}^d$, $\mathcal{K}_1, \ldots, \mathcal{K}_{2^d}$ its Klein polyhedra, $\mathcal{V}(\mathcal{K}_i)$ set of vertices

Two-dimensional statement

$$\omega(\Lambda) = \frac{1}{2} \limsup_{\substack{|\mathbf{v}| \to \infty \\ \mathbf{v} \in \mathcal{V}(\mathcal{K}_1) \cup \mathcal{V}(\mathcal{K}_2)}} \frac{\log(\det \mathrm{St}_{\mathbf{v}})}{\log |\mathbf{v}|}$$

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Question, arbitrary dimension

Is it true that

$$\omega(\Lambda)\asymp\limsup_{\substack{|\mathbf{v}|\to\infty\\ \mathbf{v}\in\bigcup_i\mathcal{V}(\mathcal{K}_i)}}\frac{\log(\det \mathrm{St}_{\mathbf{v}})}{\log|\mathbf{v}|}\ ?$$

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E. Bigushev, O.G., 2018

For $d = 3$ we have

$$\omega(\Lambda) \leqslant \frac{2}{3} \limsup_{\substack{|\mathbf{v}| \to \infty \\ \mathbf{v} \in \bigcup_i \mathcal{V}(\mathcal{K}_i)}} \frac{\log(\det \mathrm{St}_{\mathbf{v}})}{\log |\mathbf{v}|}$$

Thank you!