Minkowski question-mark function: fixed points and the derivative

Dmitry Gayfulin

Institute for Information Transmission Problems
of the Russian Academy of Sciences

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CIRM, Marseille, Luminy.

Joint work with Nikita Shulga and Igor Kan

The Minkowski question-mark function: definition

The Minkowski question-mark function is defined as follows. For an arbitrary $x \in [0,1]$ consider its continued faction expansion

$$x = [0; a_1, a_2, \ldots, a_n, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

If x is irrational, we say that

$$?(x) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\ldots+a_k-1}} \qquad (1)$$

If x is a rational number, then we replace (1) by a finite sum. One can easily see that if x is a rational number or a quadratic irrationality, then $?(x) \in \mathbb{Q}$. In fact, the inverse statement is also true.

Question-mark function and Stern-Brocot tree

Denote by F_n the $n-$th level of Stern-Brocot tree i.e.

$$F_n : \{\xi = [0; a_1, \dots, a_k] : \ a_1 + \dots + a_k = n + 1\}$$

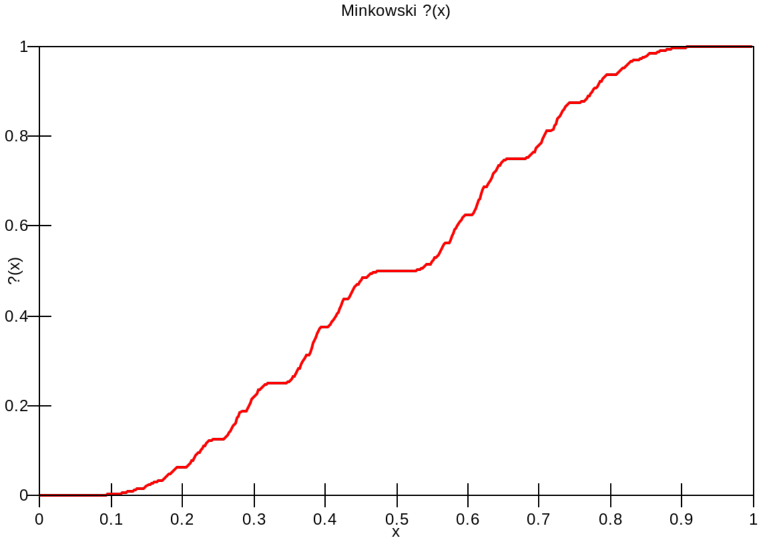
One can equivalently define the Minkowki function as the limit distribution function of sets F_n

$$?(x) = \lim_{n\to\infty} \frac{\#\{\xi \in F_n : \xi \leqslant x\}}{2^n + 1}.$$

Properties of $?(x)$

1. ?(x) is a continuous strictly increasing function, it satisfies Lipschitz condition, but is not absolutely continuous function.
2. Symmetry: $?(x) = 1 - ?(1-x)$, $?(\frac{x}{x+1}) = \frac{?(x)}{2}$
3. The derivative of ?(x) can take only two values: 0 and $+\infty$. Almost everywhere we have ?'(x)=0.
4. $?'(x) = 0$ for all $x \in \mathbb{Q}$.
5. If $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are two rationals such that $|\frac{p_1}{q_1} - \frac{p_2}{q_2}| = \frac{1}{q_1 q_2}$, then $?(\frac{p_1+p_2}{q_1+q_2}) = \frac{?(\frac{p_1}{q_1}) + ?(\frac{p_2}{q_2})}{2}$
6. $\lim\limits_{n\to\infty} \int_0^1 e^{2\pi inx} d?(x) = 0$ (Conjectured by Salem in 1943, proved by Jordan and Sahlsten in 2013)

Plot of $?(x)$



Fixed points of Minkowki function.

One can easily see that $?(0) = 0, ?(\frac{1}{2}) = \frac{1}{2}, ?(1) = 1$. As $?'(x) = 0$ at rational points, the Minkowski function has at least two more fixed points. We will call such fixed points *non-trivial*. As $?(x) = 1 - ?(1 - x)$, all fixed points are symmetric with respect to $\frac{1}{2}$ point. A folklore conjecture states that:

Conjecture

The equation $?(x) = x$ has exactly five solutions.

However, we do not even know if the number of solutions of the equation finite.

Properties of fixed points.

The computation shows that the fixed point(s) between 0 and $\frac{1}{2}$ equals ≈ 0.42037233942. If there are more then one fixed points on the interval $(0, \frac{1}{2})$, their first 4000 partial quotients in continued fraction expansion coincide.

It is easy to show that all non-trivial fixed points are irrational.

Properties of fixed points.

The computation shows that the fixed point(s) between 0 and $\frac{1}{2}$ equals ≈ 0.42037233942. If there are more then one fixed points on the interval $(0, \frac{1}{2})$, their first 4000 partial quotients in continued fraction expansion coincide.
It is easy to show that all non-trivial fixed points are irrational. However, we establish an equivalent conjecture, which deals with rational numbers only.

Conjecture

For any rational number $\frac{p}{q}$, $q > 2$ one has $|?(\frac{p}{q}) - \frac{p}{q}| > \frac{1}{2q^2}$.

So, if one shows that $\frac{p}{q}$ cannot be convergent fraction to $?(\frac{p}{q})$, he will prove that there are only 2 non-trivial roots.

Stable and unstable fixed points of $?(x)$

As $?(x)$ is a monotonic function, then for any $x \in [0,1]$ the sequence of iterations of Minkowski function

$$?^n(x) = \underbrace{?(? \ldots ?}_{n\ times}(x) \ldots)$$

is also monotonic and tends to some fixed point. We will call an isolated fixed point x_0 stable if there exists $\delta > 0$ such that $\forall x \in (x_0 - \delta, x_0 + \delta)$ one has

$$\lim_{n \to \infty} ?^n(x) = x_0.$$

We will call x_0 unstable otherwise. If X is the set of non-trivial fixed points from $(0, \frac{1}{2})$ interval, then the smallest and the biggest points of X are unstable.

Main result

Denote $\kappa_1 = \frac{\log \frac{\sqrt{5}+1}{2}}{\log \sqrt{2}} \approx 1.388$,
$\kappa_2 = \frac{4 \log \frac{5+\sqrt{29}}{2} - 5 \log (2+\sqrt{5})}{\log \frac{5+\sqrt{29}}{2} - 5 \log (2+\sqrt{5}) - \log \sqrt{2}} \approx 4.401$

Theorem (DG,N. Shulga, 2018)

Let $x = [0; a_1, ..., a_n, ..]$ be unstable fixed point of Minkowski question mark function on the interval $(0, \frac{1}{2})$. Then there exists N such that $\forall n > N$

$$a_{n+1} < (\kappa_1 - 1) \sum_{i=1}^{n} a_i. \qquad (2)$$

On the other side,

$$\sum_{i=1}^{n} a_i < \kappa_2 n \qquad (3)$$

Corollary: irrationality measure

From the previous theorem we deduce a result on irrationality measure of unstable fixed points of $?(x)$:

Corollary

Let $x = [0; a_1, \ldots, a_n, ..]$ be unstable fixed point of Minkowski question mark function on the interval $(0, \frac{1}{2})$. Then x has irrationality measure 2.
Moreover, there exists an absolute constant $c > 0$ such that

$$|x - \frac{p}{q}| > \frac{c}{q^2 \log q}$$

for any $p, q \in \mathbb{Z}$.

We recall that $?'(x)$, if exists, can take only two values - 0 and $+\infty$. Consider irrational $x = [0; a_1, ..., a_n, ..]$. Denote $S_t(x) = a_1 + \ldots + a_t$.

Theorem (A.Dushistova, N.Moshchevitin, I.Kan, 2009)

(i)Let for irrational $x \in (0,1)$ one has

$$\limsup_{t\to\infty} \frac{S_x(t)}{t} < \kappa_1$$

Then $?'(x)$ exists and $?'(x) = +\infty$.
(ii)Let for irrational $x \in (0,1)$ one has

$$\liminf_{t\to\infty} \frac{S_x(t)}{t} > \kappa_2$$

Then $?'(x)$ exists and $?'(x) = 0$.

Both constants are optimal and the theorem is non-improvable.

?'(x) with bounded partial quotients - I

We consider same problem for x having uniformly bounded partial quotients. Denote by E_n the set of all $x = [0; a_1, ..., a_n, ..]$ such that $\forall i \ a_i \leqslant n$. There exist constants κ_1^n, κ_2^n such that

Theorem (A.Dushistova, N.Moshchevitin, I.Kan, 2009)

(i)Let for irrational $x \in (0,1) \in E_n, n \geqslant 5$ one has

$$\limsup_{t\to\infty} \frac{S_x(t)}{t} < \kappa_1^n$$

Then $?'(x)$ exists and $?'(x) = +\infty$.
(ii)Let for irrational $x \in (0,1)$ one has

$$\liminf_{t\to\infty} \frac{S_x(t)}{t} > \kappa_2^n$$

Then $?'(x)$ exists and $?'(x) = 0$.

?'(x) with bounded partial quotients -II

Both constants are also optimal and

$$\lim_{n\to\infty} \kappa_1^n = \kappa_1, \quad \lim_{n\to\infty} \kappa_1^n = \kappa_1,$$

Theorem (DG, I.Kan, 2018)

(i)Suppose that $x \in E_n$, $?'(x)$ exists and equals 0. Then for t and n big enough one has

$$\max_{u\leqslant t}(S_x(u) - \kappa_1^n u) > \sqrt{t}$$

(ii)For any $n \geqslant 5$ there exists $x \in E_n$ such that $?'(x) = 0$ and for all t big enough one has

$$S_x(t) - \kappa_1^n t \leqslant (2^{\frac{2}{3}} n^{\frac{2}{3}} + 21 n^{\frac{2}{3}})\sqrt{t}$$

Main lemma and optimization problem

Denote by q_n the denominator of a convergent continued fraction $[0; a_1, \ldots, a_n]$.

Lemma (N. Moshchevitin)

Let for irrational $x \in (0,1)$ and arbitrary δ there exists a natural $t = t(x,\delta)$ such that

$$\frac{?(x+\delta)-?(x)}{\delta} \geqslant \frac{q_t q_{t-1}}{2^{S_x(t)+4}}$$

$$\frac{?(x+\delta)-?(x)}{\delta} \leqslant \frac{q_t^2}{2^{S_x(t)-2}}$$

The end

Thank you for your attention!