On the Markov Equation and Outer Automorphism of $\mathrm{PGL}(2,\mathbb{Z})$

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I.
Introduction to Markov Theory

Markov Equation: $X^2 + Y^2 + Z^2 = 3XYZ$

Integral solutions $(x, y, z) \in \mathbb{Z}_+^3$ of this equation are called Markov triples and each integer in a Markov triple is called a Markov number.

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For example, $\mathbf{1}$ is a Markov number since $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ is a solution.
There are also other solutions $(1, 89, 34), (29, 14701, 169), \ldots$

Question. How can we find all possible integer solutions?
There is a simple algorithm to produce all Markov triples.

Markov Numbers

Let (m, m_1, m_2) be a solution of Markov equation:

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

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We know that m is a solution. Denote its other solution by m'. Hence

$$m + m' = 3m_1m_2 \quad \& \quad mm' = m_1^2 + m_2^2$$

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$$\implies m' = 3m_1m_2 - m$$

Markov Numbers

By exchanging the roles of m, m_1, m_2 and starting with (m, m_1, m_2), we may derive three new solutions:

$$(m', m_1, m_2) \qquad (m, m_1', m_2) \qquad (m, m_1, m_2')$$

This gives a tree:

$$a, b, c$$

$$a, 3ab - c, b \qquad\qquad b, 3bc - a, c$$

Markov Tree

$(1,1,1)$

Markov Tree

(1,1,1)

|

(1,2,1)

Markov Tree

(1,1,1)

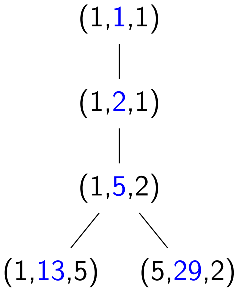
|

(1,2,1)

|

(1,5,2)

Markov Tree



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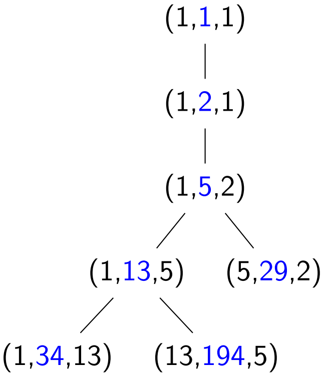

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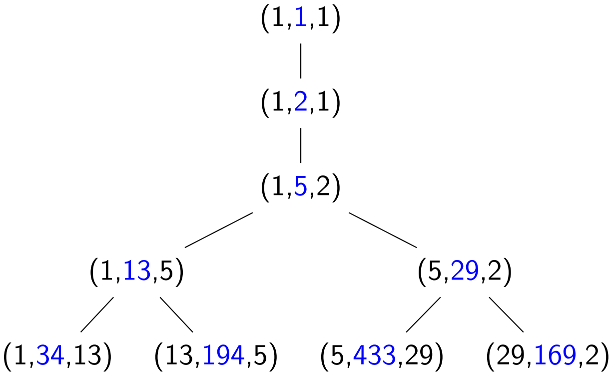
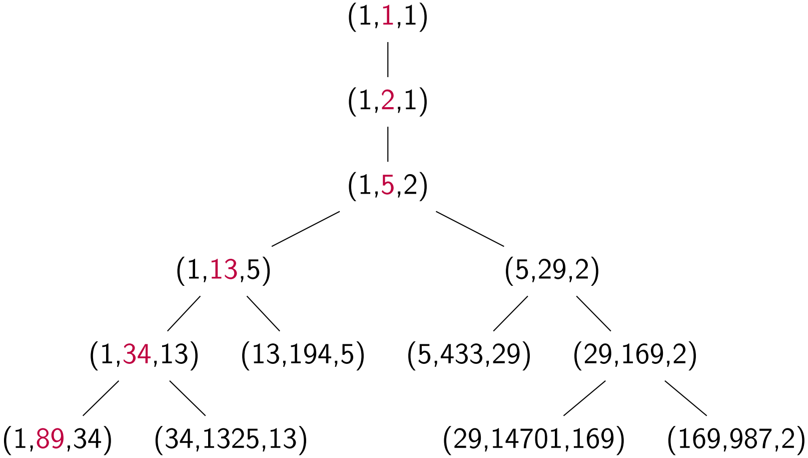


The set of maximum Markov number in the triple is:

$$\mathcal{M} = \{1, 2, 5, 13, 29, 34, \ldots\}$$

Example:

Fibonacci Number F_{2n+1} are Markov numbers.



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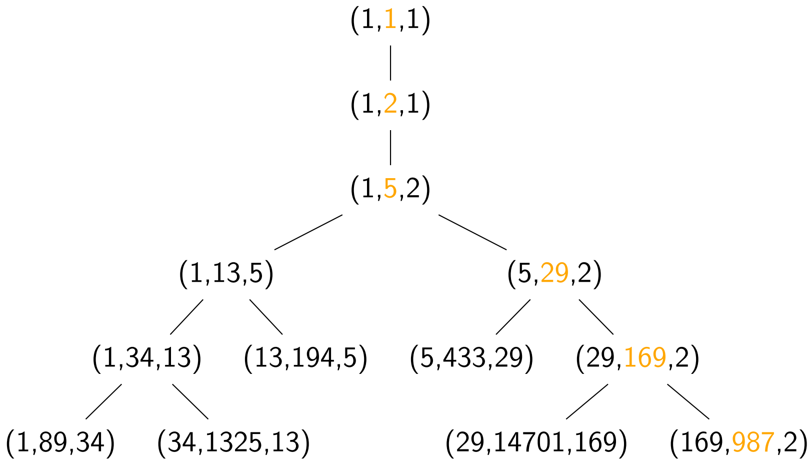

II.
Diophantine Analysis via Markov equation

Simple Continued Fractions

Every real number $\alpha \in [0,1]$ can be written in the form:

$$\alpha = [0, a_1, a_2, a_3, ...] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $a_i \in \mathbb{Z}$, $a_i > 0$ for $i > 0$ and this form is called **simple continued fraction** expansion of the real number α and $r_n = [a_0, a_1, a_2, \ldots, a_n]$ is called nth convergent of α.

Recall.

$$\mathrm{GL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}$$

Let α and β be two irrational numbers.

Definition. We say $\alpha \sim \beta$ if there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z})$ with

$$\alpha = \frac{a\beta + b}{c\beta + d}$$

$GL(2,\mathbb{Z})$ -Equivalence

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Lemma. We say $\alpha \sim \beta$ if their continued fraction expansions eventually coincide

$$\alpha = [a_0, a_1, a_2, \ldots, a_k, \gamma] \qquad \beta = [b_0, b_1, b_2, \ldots, b_l, \gamma]$$

Dirichlet's Theorem (1842)

The origin of the Markov's result is Dirichlet theorem:

Theorem. Let $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. There exists $p/q \in \mathbb{Q}$ with $q \leq N$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}$$

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$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{qN}$$

Corollary. If $\alpha \notin \mathbb{Q}$, then there exists infinitely many rational p/q with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$$

so every irrational has infinitely many "good" approximations.

Can we find "better" rational approximations ?

Roth's theorem (1955) tells us that if

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\mathbf{1}.q^{2+\varepsilon}}$$

is satisfied for infinitely many p/q with $\varepsilon > 0$ then α is **not** algebraic. So we cannot improve on the exponent 2.

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Can we improve on the constant **1** ?

Definiton. Let $\alpha \in \mathbb{R}$.

1. $L(\alpha) = \sup\left\{L > 0 : \left|\alpha - \frac{p}{q}\right| < \frac{1}{Lq^2}, \quad \textit{for infinitely many } p/q\right\}$ is called the **Lagrange number of** α.
2. $\mathcal{L} = \{L(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$ is called **Lagrange spectrum**.

How to compute Lagrange Number ?

Lemma. Let $\alpha = [a_0, a_1, a_2, \dots]$ be an irrationnal. Then

$$L(\alpha) = \lim_{n\to\infty} \sup([a_{n+1}, a_{n+2}, a_{n+3}, \dots] + [0, a_n, a_{n-1}, \dots, a_1])$$

For example,

- $L(\frac{1+\sqrt{5}}{2}) = \sqrt{5}$
- $L(1+\sqrt{2}) = \sqrt{8}$

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Remark.

$$\alpha \sim \beta \Longrightarrow L(\alpha) = L(\beta)$$

$$\underset{?}{\Longleftarrow}$$

We will see this is conjectured to be true under some hypothesis.

Hurwitz Theorem (1891)

Theorem. $\alpha \notin \mathbb{Q} \implies \exists$ inifinitely many rational p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

Moreover, if α is $GL(2,\mathbb{Z})$-equivalent to $\gamma_1 := (1+\sqrt{5})/2$, we cannot replace $\sqrt{\mathbf{5}}$ by a greater constant.

Suppose next that α is **not** $GL(2,\mathbb{Z})$-equivalent to γ_1. Then there exists infinitely many rational p/q

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{\mathbf{8}}q^2}$$

and we cannot replace $\sqrt{\mathbf{8}}$ by a greater constant when $\alpha \sim \gamma_2 = 1 + \sqrt{2}$. More generally;

Markov's Theorem (1879)

Theorem. Let $\mathcal{M} = \{1, 2, 5, 13, 29, 34, \dots\}$ be the set of Markov numbers. Then

i. The **Lagrange spectrum below 3** is given by the set

$$\mathcal{L}_{<3} = \left\{ \frac{\sqrt{9m^2 - 4}}{m} : m \in \mathcal{M} \right\}$$

There is a sequence of inequivalent quadratic irrationals

$$\gamma_m = \frac{m + 2u + \sqrt{9m^2 - 4}}{2m}$$

where u is the characteristic number of m and whose Lagrange numbers are

$$L(\gamma_m) = \frac{\sqrt{9m^2 - 4}}{m}$$

ii. Conversely, every $L(\alpha) < 3$ with $\alpha \notin \mathbb{Q}$ is of this form.

Uniqueness Conjecture (1913)

Uniqueness Conjecture. Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Then

$$L(\alpha) = L(\beta) < 3 \Longrightarrow \alpha \sim \beta$$

III.
$\mathrm{Out}(\mathrm{PGL}(2,\mathbb{Z}))$ and Markov Theory

Functional Equation I

$$\begin{cases} g(x+1) = g(x) + g(\frac{1}{x}) \\ g(x) = g(\frac{x}{x+1}) \end{cases}$$

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$$\Downarrow$$

$$g : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$$
$$\frac{p}{q} \mapsto p$$

This function is called numerator and denoted by $\mathrm{num}(x)$.

Functional Equation II :

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$$f(n) = F_{n+1}$$

where F_n is n th Fibonacci number with $n \in \mathbb{N}$.

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where F_n is n th Fibonacci number with $n \in \mathbb{N}$.

Can we extend f on $\mathbb{Q}$?

Yes. It is called the conumerator and denoted by $\mathrm{con}(x)$.

The answer is as follows:

Lemma. For all $x \in \mathbb{Q}_{>0}$ and $n \in \mathbb{N}$, we have:

$$\operatorname{con}(n + x) = F_n \operatorname{con}(x) + F_{n+1} \operatorname{con}(1/x)$$

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$$\mathrm{con}(n+x) = F_n \mathrm{con}(x) + F_{n+1} \mathrm{con}(1/x)$$

For example,

$$\begin{aligned} \mathrm{con}(3/2) &= \mathrm{con}(1+1/2) \\ &= F_1 \mathrm{con}(1/2) + F_2 \mathrm{con}(2) \\ &= F_1 + F_2 F_3 \\ &= 3 \end{aligned}$$

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It can be shown that

$$\mathrm{con}\left(\frac{\mathrm{con}(x)}{\mathrm{con}(1/x)}\right) = \mathrm{num}(x)$$

Involution of irrationals : Jimm

Jimm is defined as follows:

$$J(x) := \frac{\mathrm{con}(x)}{\mathrm{con}(1/x)}$$

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By using functional equation of conumerator

$$J(1+x) = 1 + 1/J(x) \quad \& \quad J(1/x) = 1/J(x)$$

$\Longrightarrow$ Jimm admits a natural extension to $\mathbb{Q}$.

Jimm is an involution

Recall. $J(x) = \dfrac{\mathrm{con}(x)}{\mathrm{con}(1/x)}$

Then,

$$J(J(x)) =$$

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Recall. $J(x) = \dfrac{\mathrm{con}(x)}{\mathrm{con}(1/x)}$

Then,

$$J(J(x)) = \frac{\mathrm{con}\left(\dfrac{\mathrm{con}(x)}{\mathrm{con}(1/x)}\right)}{\mathrm{con}\left(\dfrac{\mathrm{con}(1/x)}{\mathrm{con}(x)}\right)} = \frac{\mathrm{num}(x)}{\mathrm{num}(1/x)} = x$$

$$\Rightarrow \boldsymbol{J(J(x)) = x}$$

Jimm is defined on $\mathbb{R} \setminus \mathbb{Q}$

We can express Jimm of irrational in terms of continued fraction:
Lemma.

$$J([a_0, a_1, a_2, a_3, \dots]) = \mathbf{[1_{a_0-1}, 2, 1_{a_1-2}, 2, 1_{a_2-2}, 2, \dots]}$$

with two rules:

$$[\dots, a, 1_0, b, \dots] := [\dots, a, b, \dots]$$
$$[\dots, a, 1_{-1}, b, \dots] := [\dots, a+b-1, \dots]$$

where 1_k representes the sequence $1, 1, \dots, 1$ of length k.

Origin of Jimm:

Recall. $\mathrm{PGL}(2,\mathbb{Z}) = \left\{ \frac{px+q}{rx+s} : p, s, q, r \in \mathbb{Z},\ ps - qr = \pm 1 \right\}$

Dyer (1978) showed the existence of a unique outer automorphism of $\mathrm{PGL}(2,\mathbb{Z})$.

M. Uludağ & H. Ayral (2015) reformulated action on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, explicitely on continued fraction expansion:

$$J(x)$$

What relationship with the theory of Markov?

Theorem. Jimm preserves the set of quadratic irrational numbers.

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Theorem. Jimm preserves the set of quadratic irrational numbers.

Question. What is $J(\gamma_m)$ where γ_m is a Markov irrational ?

Markov irrationals γ_m			$J(\gamma_m)$
$\gamma_1 = \dfrac{1+\sqrt{5}}{2}$	$= [\bar{1}]$	$\longrightarrow$	∞
$\gamma_2 = 1+\sqrt{2}$	$= [\bar{2}]$	$\longrightarrow$	$[1, \overline{2}]$
$\gamma_5 = \dfrac{9+\sqrt{221}}{10}$	$= [\overline{2_2, 1_2}]$	$\longrightarrow$	$[1, \overline{2, 4}]$
$\gamma_{13} = \dfrac{23+\sqrt{1517}}{26}$	$= [\overline{2_2, 1_4}]$	$\longrightarrow$	$[1, \overline{2, 6}]$

What is the quadratic form of Jimm of Markov irrationals ?

Markov irrationals γ_m			$J(\gamma_m)$	
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$\gamma_{13} = \frac{23+\sqrt{1517}}{26}$	$= [\overline{2_2, 1_4}]$	$\longrightarrow$	$[1, \overline{2, 6}]$	$=?$

How to find quadratic form ?

The general functional equation is

$$\boxed{J(Mx) = J(M)J(x) \text{ for all } \; M \in PGL(2,\mathbb{Z})}$$

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The general functional equation is

$$\boxed{J(Mx) = J(M)J(x) \text{ for all } \; M \in PGL(2,\mathbb{Z})}$$

If there is a matrix M such that

$$x = Mx$$

then

$$J(x) = J(Mx) = J(M)J(x)$$

It means that $J(x)$ is fixed also by $J(M)$.

$$\Longrightarrow \text{Find } J(M)$$

We may express

$$J : PGL_2(\mathbb{Z}) \to PGL_2(\mathbb{Z})$$
$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mapsto J(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of conumerator function. We know that $J(1) = 1$ and $J(2) = 2$,

$$J(M1) = J(M)J(1) \Longrightarrow J\left(\frac{p+q}{r+s}\right) = \frac{a+b}{c+d}$$
$$J(M2) = J(M)J(2) \Longrightarrow J\left(\frac{2p+q}{2r+s}\right) = \frac{2a+b}{2c+d}$$

It is equivalent to say

$$\mathrm{con}\left(\frac{p+q}{r+s}\right) = a+b$$
$$\mathrm{con}\left(\frac{r+s}{p+q}\right) = c+d$$
$$\mathrm{con}\left(\frac{2p+q}{2r+s}\right) = 2a+b$$
$$\mathrm{con}\left(\frac{2r+s}{2p+q}\right) = 2c+d$$

Find a, b, c, d and put them into the matrix $J(M)$:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

$$\Big\downarrow J$$

$$\begin{pmatrix} \mathrm{con}\left(\dfrac{2p+q}{2r+s}\right) - \mathrm{con}\left(\dfrac{p+q}{r+s}\right) & 2\mathrm{con}\left(\dfrac{p+q}{r+s}\right) - \mathrm{con}\left(\dfrac{2p+q}{2r+s}\right) \\ \mathrm{con}\left(\dfrac{2r+s}{2p+q}\right) - \mathrm{con}\left(\dfrac{r+s}{p+q}\right) & 2\mathrm{con}\left(\dfrac{r+s}{p+q}\right) - \mathrm{con}\left(\dfrac{2r+s}{2p+q}\right) \end{pmatrix}$$

Recall that Markov quadratic irrational is of the form

$$\gamma_m = \frac{m + 2u + \sqrt{9m^2 - 4}}{2m}$$

and it is fixed by the matrix

$$M = \begin{pmatrix} 2m + u & 2m - u - v \\ m & m - u \end{pmatrix}$$

which is hyperbolic where m is a Markov number and u, v are characteristic numbers of associated Markov triple.

Remark. This matrix comes from a Cohn matrix.

Example

Let $m = 2$, $u = 1$, $v = 1$ and quadratic irrational associated is

$$\gamma_2 = 1 + \sqrt{2}$$

and it is fixed by the matrix M_2

$$M_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \longmapsto J(M_2) = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

$$\frac{3x+4}{2x+3} = x \Longrightarrow x = \pm\sqrt{2}$$

$\Longrightarrow J(\gamma_2) = \sqrt{2}$

m	Markov irrational γ_m	$J(\gamma_m)$
1	$(1+\sqrt{5})/2$	∞
2	$1+\sqrt{2}$	$\sqrt{2}$
5	$(9+\sqrt{221})/10$	$\sqrt{6}-1$
13	$(23+\sqrt{1517})/26$	$\sqrt{12}-2$
29	$(53+\sqrt{7565})/58$	$(\sqrt{210}-6)/6$
34	$(15+\sqrt{650})/17$	$\sqrt{20}-3$
89	$(157+\sqrt{71285})/178$	$\sqrt{30}-4$
169	$(309+\sqrt{257045})/338$	$(\sqrt{7140}-35)/35$
194	$(344+\sqrt{338720})/388$	$\sqrt{119}-2$
233	$(411+\sqrt{488597})/466$	$\sqrt{42}-5$
433	$(791+\sqrt{1687397})/866$	$(12\sqrt{143}-60)/59$

Remark that $J(\gamma_m)$ is simpler !

m	Markov irrational γ_m	$J(\gamma_m)$
1	$(1+\sqrt{5})/2$	∞
2	$1+\sqrt{2}$	$\sqrt{2}$
5	$(9+\sqrt{221})/10$	$\sqrt{6}-1$
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Theorem. (E. , Uludağ, 2018)

Let γ_m be a Markov irrational with $m = F_{2n+1}$ as follows

$$\gamma_{F_{2n+1}} = \frac{F_{2n+1} + F_{2n-1} + \sqrt{9F_{2n+1}^2 - 4}}{2F_{2n+1}}$$

fixed by the hyperbolic matrix

$$M = \begin{pmatrix} 2F_{2n+1} + F_{2n-1} & F_{2n+2} - F_{2n-3} \\ F_{2n+1} & F_{2n} \end{pmatrix}$$

Then image of M under Jimm is

$$J(M) = \begin{pmatrix} 3 & 6n-2 \\ 2 & 4n-1 \end{pmatrix}$$

and positive fixed point of this matrix which is equal to $J(\gamma_m)$ is

$$\sqrt{n^2+n} - n + 1$$

Thanks for your attention !