On the distance to the nearest square-free polynomial

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Definitions

For an integer polynomial

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

of degree $d \geq 1$, its *length* $L(f)$ is defined by

$$L(f) = |a_d| + |a_{d-1}| + \cdots + |a_0|.$$

and its *height* $H(f)$ by

$$H(f) = \max\{|a_d|, |a_{d-1}|, \ldots, |a_0|\}.$$

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Turán's problem

In 1960s, Turán asked the following: does there exist an absolute constant C such that for any polynomial $f \in \mathbb{Z}[x]$ there is an irreducible (over the rational numbers) polynomial $g \in \mathbb{Z}[x]$ of degree at most $\deg f$ satisfying

$$L(f - g) \leq C?$$

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Results on Turán's problem

In 1970, Schinzel proved that $C = 3$ suffices if one removes the condition on the degree of g. More precisely, he showed that if $f \in \mathbb{Z}[x]$ is of degree d then there are infinitely many irreducible polynomials $g \in \mathbb{Z}[x]$ such that

$$L(f - g) \leq \begin{cases} 2 & \text{if } f(0) \neq 0, \\ 3 & \text{always,} \end{cases}$$

and, moreover, at least one of them satisfies

$$\deg g \leq \exp((5d + 7)(\|f\| + 3)),$$

where $\|f\|$ stands for the sum of the squares of the coefficients of f.

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In 2010 (Acta Arith.), Banerjee and Filaseta improved the above upper bound to

$$\deg g \leq 8 \max\{d + 3, c_0\} 5^{8\|f\|+9},$$

where c_0 is an effectively computable absolute constant. In addition, using computational strategies, it has been confirmed (Bérczes, Hajdu, Filaseta, Mossinghoff, Lee, Ruskey, Williams) that if $f \in \mathbb{Z}[x]$ has degree $d \leq 40$ then there exists an irreducible polynomial $g \in \mathbb{Z}[x]$ with $\deg g = d$ and $L(f - g) \leq 5$.

On the other hand, although the trivial example $f(x) = x^3$ shows that $C \geqslant 2$, it is not known that the optimal constant C should be strictly greater than 2.

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A version of Turán's problem

Let us consider a variant of Turán's problem, where "irreducible polynomial g" is replaced by "square-free polynomial g".

For this, we pose the following conjecture:

Conjecture 1

For any $f \in \mathbb{Z}[x]$ of degree d, there is a square-free polynomial $g \in \mathbb{Z}[x]$ of degree at most d satisfying

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Problem of Szegedy

Another problem related to Turán's problem is that of Szegedy: is there a constant C_0 depending only on d such that for any $f \in \mathbb{Z}[x]$ of degree d the polynomial $f(x) + t$ is irreducible for some $t \in \mathbb{Z}$ with $|t| \leqslant C_0$.

In general, the problem of Szegedy is still open, although there are some partial results of Győry, Bérczes and Hajdu. However, in our setting, when"irreducible" is replaced by "square-free", this problem becomes very simple. One can take, for instance, $C_0 = \lfloor d/2 \rfloor$.

Theorem 2

For any $f \in \mathbb{Z}[x]$ of degree d, at least one of the polynomials $f(x) + t$, where $t \in \mathbb{Z}$ satisfies $|t| \leqslant \lfloor d/2 \rfloor$, is square-free.

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Problem of Szegedy

Another problem related to Turán's problem is that of Szegedy: is there a constant C_0 depending only on d such that for any $f \in \mathbb{Z}[x]$ of degree d the polynomial $f(x) + t$ is irreducible for some $t \in \mathbb{Z}$ with $|t| \leqslant C_0$.

In general, the problem of Szegedy is still open, although there are some partial results of Győry, Bérczes and Hajdu. However, in our setting, when"irreducible" is replaced by "square-free", this problem becomes very simple. One can take, for instance, $C_0 = \lfloor d/2 \rfloor$.

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Let S be a subset of $\mathbb{Z}$ with the property that for each integer $t \in S$ some h_t^2, where $h_t \in \mathbb{Z}[x]$ is of degree at least 1, divides the polynomial $f(x) + t$. Then, $h_t \neq h_s$ when $t \neq s$ both belong to S, since otherwise $h_t \mid (t - s)$, a contradiction. Also, h_t divides the derivative f' for every $t \in S$, so the cardinality of the set S does not exceed $\deg f' \leqslant d - 1$. The assertion of the theorem now follows, because the set $\{-\lfloor d/2 \rfloor, \dots, 0, \dots, \lfloor d/2 \rfloor\}$ contains at least d integers. $\square$

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Is the constant 2 in Conjecture 1 optimal?

In general, the condition $L(f-g) \leq 2$ of Conjecture 1 cannot be relaxed.

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For any integer $d \geq 15$, there exist infinitely many polynomials $f \in \mathbb{Z}[x]$ of degree d such that each polynomial $g \in \mathbb{Z}[x]$ satisfying

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One example of such degree 15 polynomials is

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Its root 0 has multiplicity 2. Also, 0 is a root of multiplicity 2 of any polynomial $f(x) \pm x^k$, where $k \geqslant 2$ is an integer, whereas $1/2, -1/2, 1/6$ and $-1/6$ are multiple roots of $f(x) - x, f(x) + 1, f(x) + x$ and $f(x) - 1$, respectively.

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The next result is a weak form of Conjecture 1 by relaxing the condition on the degree of g:

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For any $f \in \mathbb{Z}[x]$ of degree d and any integer

$$n > L(f'), \tag{1}$$

there is a square-free polynomial $g \in \mathbb{Z}[x]$ satisfying $\deg g = n$ *and*

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In the following theorem, we establish the existence of one square-free polynomial close to f but of degree that for large $L(f)$ can be much smaller than the bound in (1). (In terms of $L(f)$, the bound $dL(f)$ on $\deg g$ is replaced by the bound $2.2d(\log d/\log\log d)^3 \log L(f)$.)

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Let $\mathbb{F}_2$ denote the binary field. For any polynomial $f \in \mathbb{F}_2[x]$ we define its *length* $L_2(f)$ to be the number of its monomials.

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For each polynomial $f \in \mathbb{F}_2[x]$ of degree $d \leq 36$ which is not square-free and satisfies $f(0) \neq 0$ there exists an integer n with $0 < n < d$ such that $x^n + f(x)$ is square-free.

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Note that from these results one can obtain various classes of polynomials $f \in \mathbb{Z}[x]$ such that there exists a square-free polynomial $g \in \mathbb{Z}[x]$ of degree $\deg f$ satisfying $L(f-g) \leq 2$.

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Polynomials over prime fields

Let $\mathbb{F}_p$ be the finite field with p elements, where p is a prime number. For any polynomial $f \in \mathbb{F}_p[x]$, define its *length* $L_p(f)$ by choosing each of its coefficients in the interval $(-p/2, p/2]$ and then summing their absolute values (in $\mathbb{Z}$).

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As an analogue of Conjecture 1, we pose the following question:

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Does for any prime number p and any polynomial $f \in \mathbb{F}_p[x]$ of degree d there exist a square-free polynomial $g \in \mathbb{F}_p[x]$ of degree at most d satisfying

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Proof of Theorem 3 (example with large coefficients)

For any polynomial $f \in \mathbb{Z}[x]$ of the form $x^2 h(x)$ with non-zero $h(x) \in \mathbb{Z}[x]$ (so that f automatically is not square-free), if there were a square-free polynomial $g \in \mathbb{Z}[x]$ satisfying $L(f - g) \leq 1$, then g must be of the form $f(x) \pm 1$ or $f(x) \pm x$.

So, our purpose is to find polynomials $f \in \mathbb{Z}[x]$ of the form $x^2 h(x)$ such that none of the following four polynomials

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is square-free.

Proof of Theorem 3 (example with large coefficients)

For any polynomial $f \in \mathbb{Z}[x]$ of the form $x^2 h(x)$ with non-zero $h(x) \in \mathbb{Z}[x]$ (so that f automatically is not square-free), if there were a square-free polynomial $g \in \mathbb{Z}[x]$ satisfying $L(f - g) \leq 1$, then g must be of the form $f(x) \pm 1$ or $f(x) \pm x$.

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Proof of Theorem 3

Assume that

$$\begin{aligned} &f \equiv 0 \pmod{x^2}, \quad f \equiv -1 \pmod{(2x+1)^2},\\ &f \equiv x \pmod{(2x-1)^2}, \quad f \equiv 1 \pmod{(6x+1)^2},\\ &f \equiv -x \pmod{(6x-1)^2}. \end{aligned} \tag{2}$$

Then, all the solutions in $\mathbb{Z}[x]$ of (2) meet our purpose.

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Proof of Theorem 3

By the Chinese Remainder Theorem and using PARI/GP, we obtain a solution $f_0 \in \mathbb{Q}[x]$ of (2):

$$\begin{aligned} f_0(x) = & 106515x^9 - 8991x^8 - \frac{236133}{4}x^7 + \frac{20385}{4}x^6 \\ & + \frac{152209}{16}x^5 - \frac{13701}{16}x^4 - \frac{22207}{64}x^3 + \frac{2243}{64}x^2. \end{aligned}$$

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Proof of Theorem 3

Then,

$$h(x) = 20736x^{10} - 11520x^8 + 1888x^6 - 80x^4 + x^2.$$

So, the general solution of (2) in $\mathbb{Q}[x]$ has the form

$$f = f_0 + hf_1, \quad f_1 \in \mathbb{Q}[x].$$

Now, we want to choose suitable f_1 such that $f \in \mathbb{Z}[x]$.

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Notice that f_0 has six coefficients not in $\mathbb{Z}$. We then choose f_1 to be a polynomial in $\mathbb{Q}[x]$ of degree 5:

$$f_1(x) = a_5x^5 + \cdots + a_1x + a_0$$

such that $f_0 + hf_1 \in \mathbb{Z}[x]$, that is, hf_1 is congruent to $-f_0$ modulo the integers. By comparing the coefficients modulo the integers starting from the lowest term, we obtain

$$a_0 \in \frac{61}{64} + \mathbb{Z}, \quad a_1 \in \frac{63}{64} + \mathbb{Z}, \quad a_2 \in \frac{9}{16} + \mathbb{Z},$$
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$$f(x) + k(2x+1)^2(2x-1)^2(6x+1)^2(6x-1)^2 x^{d-8},$$

where k is any non-zero integer. Then, by the construction of $f(x)$, we complete the proof for $d \geq 16$.

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Publication

The results are joint with Min Sha (Macquarie University, Sydney) and will appear in Acta Arithmetica.

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