The sum-of-digits function, Primes, and Uniform Distribution modulo 1

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Summary

Distribution of ...

- ★ q-ary sum-of-digits function
- ★ q-ary sum-of-digits function of **primes**
- ★ q-ary sum-of-digits functions with respect to **2 bases**
- ★ q-ary sum-of-digits functions of **primes** with respect to **2 bases**
- ★ Zeckendorf sum-of-digits function
- ★ Zeckendorf sum-of-digits function of **primes**

★ Digital Expansions

q-ary digital expansion:

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}$$

Sum-of-digits function:

$$\boxed{s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)}$$

Strongly q-additive function:

$$f(n) = \sum_{i=0}^{\ell-1} f(\varepsilon_i(n)) \qquad (f(0) = 0)$$

★ Digital Expansions

Theorem

Suppose that $f(n)$ is a non-zero and integer valued strongly q-additive function.
Then the sequence $\boxed{(\alpha f(n))}$ *is* **uniformly distributed modulo 1** *if and only if* α **is irrational**.

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Proof follows from

$$\sum_{n<q^L} e(h\alpha f(n)) = (1 + e(h\alpha f(1)) + e(h\alpha f(2)) + \cdots + e(h\alpha f(q-1)))^L,$$

and Weyl's criterion ($e(x)$ denotes $\boxed{e(x) = e^{2\pi i x}}$).

★ Digital Expansions

- ★ Central limit theorem
- ★ Local limit law
- ★ Uniform distribution modulo 1
- ★ Uniform distribution modulo *m*

★ Digital Expansions

★ Central Limit Theorem

$$\# \left\{ n < N : f(n) \leqslant \mu \log_q N + t\sigma \sqrt{\log_q N} \right\} = N\,\Phi(t) + o(N)$$

where

$$\mu = \frac{1}{q} \sum_{\ell=0}^{q-1} f(\ell) \qquad \sigma^2 = \frac{1}{q} \sum_{\ell=0}^{q-1} (f(\ell) - \mu)^2$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\, du.$$

★ Digital Expansions

★ Local Limit Theorem

$$\#\{n < N : f(n) = k\} = \frac{N}{\sqrt{2\pi\sigma^2 \log_q N}} \exp\left(-\frac{(k-\mu \log_q N)^2}{2\sigma^2 \log_q N}\right) (1 + o(1)$$

provided that $\gcd(f(1), \ldots, f(q-1)) = 1$.

★ Digital Expansions

★ Uniform distribution modulo 1

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★ Uniform distribution modulo m

$\boxed{f(n)}$ is uniformly distributed modulo m if and only if $\gcd(f(1), \ldots, f(q-1), m) = 1$:

$$\#\{n < N : f(n) \equiv \ell \bmod m\} \sim \frac{N}{m}.$$

★ Digital Expansions

Generating function

$$\sum_{n<q^L} x^{f(n)} = \sum_{k\geqslant 0} \#\{n < q^L : f(n) = k\} \cdot x^k = \left(1 + x^{f(1)} + \cdots + x^{f(q-1)}\right)^L$$

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- ★ $x = e^{it/\sqrt{\log N}}$ + Levy's theorem $\rightarrow$ Central limit theorem
- ★ $x = e^{it}$ + saddle point method $\rightarrow$ Local limit law
- ★ $x = e^{2\pi i\alpha}$ + Weyl's criterion $\rightarrow$ Uniform distribution modulo 1
- ★ $x = e^{2\pi ij/m}$ + discrete Fourier analysis $\rightarrow$ Unif. distr. mod m

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Remark: In all cases we need $|x| = 1$, that is, $x = e^{2\pi i\alpha} = e(\alpha)$.

★ Digital Expansions of Primes

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$$\boxed{f(p),\ p \in \mathbb{P},\ p < N},$$

where $f(n)$ is strongly q-additive.

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Remark: Everything can be generalized to integer valued **strongly** q**-additive** functions.

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$$\sum_{p<N} x^{f(p)} \quad \text{or} \quad \sum_{n<N} \Lambda(n) x^{f(n)}$$

for $x = e(\alpha)$.
($\Lambda(n)$ ... Von Mangoldt function, $\Lambda(p^k) = \log p$ and 0 otherwise)

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Main issue:

$$\boxed{\sum_{n<N} \Lambda(n) e(\alpha f(n)) = o(N)}$$

for non-integers α.

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Problem: No product representation!!

★ Orthogonality to arithmetic functions

$h(n)$ ... proper arithmetic function ($\Lambda(n)$, $\mu(n)$, $\lambda(n)$, ...):

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★ Sarnak Conjecture:
$a(n) = f(T^n x_0)$ ... deterministic sequence
[(X, T) dyn. system with zero entropy, f cont.]

$$?? \quad \boxed{\sum_{n<N} \mu(n)a(n) = o(N)} \quad ??$$

★ Orthogonality to arithmetic functions

Theorem (Mauduit+Rivat 10, Martin+Mauduit+Rivat 14,15)

For every integer valued strongly q-additive function $f(n)$ *with* $\gcd(f(1),\ldots,f(q-1),q-1)=1$ *and for and for real* α *with* $d\alpha \notin \mathbb{Z}$*, where* $d=\gcd(f(2)-2f(1),\ldots,f(q-1)-(q-1)f(1),q-1)$*, we have*

$$\sum_{n<N} \Lambda(n) e(\alpha f(n)) = o(N)$$

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$$\sum_{n<N}\Lambda(n)e(\alpha f(n))=o(N)$$

Theorem (Extension of Dartyge-Tenenbaum 2005)

For every integer valued strongly q-additive function $f(n)$ with $\gcd(f(1),\ldots,f(q-1),q-1)=1$, for every **multiplicative function** *$h(n)$ with $|h(n)|\leqslant 1$ and the property $\sum_{n<N}h(n)=o(N)$, and for real α with $d\alpha\notin\mathbb{Z}$, we have*

$$\sum_{n<N}h(n)e(\alpha f(n))=o(N)$$

★ Digital Expansions with Respect to 2 Bases

$\boxed{\boxed{\gcd(q_1, q_2) = 1}}$, $\boxed{f(n) = s_{q_1}(n)}$, $\boxed{g(n) = s_{q_2}(n)}$.

What can be said about the common distribution of

$$\boxed{(f(n), g(n)),\ n < N} \quad ?$$

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$$\sum_{n<N} x^{f(n)} y^{g(n)} = \sum_{k_1,k_2} \#\{n < N : f(n) = k_1,\, g(n) = k_2\} \cdot x^{k_1} y^{k_2}$$

for $x = e(\alpha)$, $y = e(\beta)$

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Main issue:

$$\boxed{\sum_{n<N} e(\alpha f(n) + \beta g(n)) = o(N)}$$

for non-integers α, β.

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Problem: No product representation!!

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★ Central Limit Theorem

$$\#\left\{n < N : s_{q_j}(n) \leqslant \mu_j \log_{q_j} N + t\sigma\sqrt{\log_{q_j} N}, j = 1, 2\right\}$$
$$= N\,\Phi(t_1)\Phi(t_2) + o(N)$$

where

$$\mu_j = \frac{q_j - 1}{2} \qquad \sigma_j^2 = \frac{q_j^2 - 1}{12}$$

and

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\,du.$$

★ Digital Expansions with Respect to 2 Bases

★ Local Limit Theorem

$$\frac{1}{N}\#\{n < N \,|\, s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\}$$
$$= d \prod_{\ell=1}^{2} \left(\frac{1}{\sqrt{2\pi \frac{q_\ell^2-1}{12} \log_{q_\ell} N}} \exp\left(- \frac{\left(k_\ell - \frac{q_\ell-1}{2}\log_{q_\ell} N\right)^2}{2\frac{q_\ell^2-1}{12}\log_{q_\ell} N} \right)\right) (1 + +o$$

uniformly for all integers $k_1, k_2 \geqslant 0$ with $k_1 \equiv k_2 \bmod d$, where $d = \gcd(q_1 - 1, q_2 - 1)$

★ Digital Expansions with Respect to 2 Bases

★ Uniform distribution modulo 1

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★ Uniform distribution modulo m

Suppose that $\gcd(q_1 - 1, m_1) = \gcd(q_2 - 1, m_2) = 1$. Then

$$\#\{n < N : s_{q_1}(n) \equiv \ell_1 \bmod m_1,\ s_{q_2}(n) \equiv \ell_2 \bmod m_2\} \sim \frac{N}{m_1 m_2}.$$

★ Digital Expansions of Primes with Respect to 2 Bases

Distribution of

$$\boxed{(f(p), g(p)),\ p \in \mathbb{P},\ p < N},$$

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for non-integers α, β.

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★ Digital Expansions of Primes with Respect to 2 Bases

Proof methods

- **Vaughan's method** (sums of type I and type II)
- Van-der-Corput inequality
- **Fourier techniques**
- Diophantine approximation techniques (**Baker's theorem**)
- Exponential sum estimates

★ Digital Expansions of Primes with Respect to 2 Bases

Theorem

Suppose that $(q_1 - 1)\alpha \notin \mathbb{Z}$ *and* $(q_1 - 1)\beta \notin \mathbb{Z}$. *Then*

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Theorem

For all $\alpha, \beta \in \mathbb{R}$ *we have*

$$\sum_{n<N} \mu(n) e(\alpha s_{q_1}(n) + \beta s_{q_2}(n)) = o(N).$$

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Theorem

The sequence $\boxed{(\alpha s_{q_1}(p) + \beta s_{q_2}(p))}$, $p \in \mathbb{P}$, *is uniformly distributed modulo 1 if and only if* α *or* β *is irrational.*

★ Zeckendorf Expansions

$F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1}$ Fibonacci numbers

$$\boxed{n = \sum_{j \geqslant 2} \varepsilon_{Z,j}(n) F_j}, \quad \varepsilon_{Z,j}(n) \in \{0,1\},\ \varepsilon_{Z,j}(n)\varepsilon_{Z,j+1}(n) = 0.$$

Zeckendorf Sum-of-digits function

$$\boxed{s_Z(n) = \sum_{j \geqslant 2} \varepsilon_{Z,j}(n)}$$

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What can be said about the distribution of

$$\boxed{s_Z(n),\ n < N} \quad ?$$

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- ★ Uniform distribution modulo m [Grabner+Tichy 91]

★ Zeckendorf Expansions of Primes

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for non-integers α.

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Theorem (D.+Müllner+Spiegelhofer 18)

We have for non-integers α and for all **multiplicative functions** $h(n)$ *with* $|h(n)| \leqslant 1$

$$\sum_{n<N} h(n) e(\alpha s_Z(n)) = o(N).$$

★ Zeckendorf Expansions of Primes

Proof method

- **Kátai method**: $\sum_{n<N} e(\alpha(s_Z(pn) - s_Z(qn))) = o(N)$ is sufficient
- The function $f(n) = s_Z(pn) - s_Z(qn)$ is **quasi-additive**
- Generating functions for quasi-additive functions
- Combinatorial lemma:
  $\exists n_1, n_2 \,:\, s_Z(pn_1) \equiv s_Z(qn_1) \bmod m,\ s_Z(pn_2) \not\equiv s_Z(qn_2) \bmod m$

Thank you very much for your attention!