RANDOM WALKS ON THE CIRCLE AND DIOPHANTINE APPROXIMATION

Joint results with Bence Borda

Sums of independent random variables 1711-

Weak law of large numbers (J. Bernoulli 1711)

Central limit theorem (de Moivre 1733)

Strong law of large numbers (Borel 1909)

Law of the iterated logarithm (Khinchin 1924)

Sums of i.i.d. random variables mod 1

$S_n = X_1 + \ldots + X_n, \quad Z_n = \{S_n\}$

Motivation:

Benford's law: leading digit of numbers in large databases is NOT uniformly distributed: $\quad P(\{1\}) = \log_{10} 2 > 0.3$

"Typical" behavior of discrepancy of $\{n_k \alpha\}$

$\{S_n\}$ is random walk on circle $\Longrightarrow$ Markov chain

(a) X_1 absolutely continuous $\Longrightarrow$ exponential convergence to uniform distribution

(b) X_1 is lattice distributed with values $k\alpha$, $(k = 0, \pm 1, \pm 2, \ldots)$, α irrational

Countable Markov chain $\Longrightarrow$ Convergence to uniform distribution is much slower

Random walks on finite groups Kesten, Diaconis, Saloff-Coste (1980-2000)

Card mixing: How many shuffles to uniformity?

Aldous (1983): Cutoff at $\frac{3}{2}\log_2 n$ steps

New York Times (January 9, 1990): *In Shuffling Cards, 7 Is Winning Number*

Random walk on circle: Moving forward or backward with angle $\pm\alpha$, α irrational

Convergence speed depends on **rational approximation properties of** α

$S_n = k\alpha$, $|k| \le n$ Assume $\alpha = \frac{p}{q} + O(q^{-100})$ $S_n = k\frac{p}{q} + O(kq^{-100})$

Schatte (1984-91)

Su (1998): If α is quadratic irrational $(\alpha = r + s\sqrt{t})$, then

$$C_1 n^{-1/2} \le \sup_x |P(S_n < x) - x| \le C_2 n^{-1/2}$$

Quadratic irrationals: bad rational approximation (worst case: $\frac{\sqrt{5}+1}{2}$)

Nonrandom analogue: $x_n = \{n\alpha\}$, α irrational

Empirical measure

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I_{(0,x)}(x_k)$$

$$D_N(x_k) = \sup_{0 \le a \le 1} \left| \frac{1}{N} \sum_{k=1}^{N} I_{[0,a)}(x_k) - a \right| \quad \textbf{discrepancy}$$

The magnitude of $D_N(\{k\alpha\})$ depends on the continued fraction digits of α

Diophantine conditions in analysis

Poincaré (1890) Planetary motion

$$\sum_{m,n\neq 0} a_{m,n}\frac{e^{i(m\omega_1+n\omega_2)t}}{m\omega_1+n\omega_2}.$$

Jupiter $\omega_1 = 299.1$", Saturn $\omega_2 = 12.5$", $2\omega_1 - 5\omega_2 \approx 0$.

Settled by **Arnold (1963)**

Siegel (1942) Stability of fix point algorithms depends on rational approximation of α in $z_0 = e^{2\pi i\alpha}$

Diophantine type

$$\sup\left\{c : \left|\alpha - \frac{p}{q}\right| < \frac{A}{q^{c+1}} \text{ for infinitely many } p/q\right\}$$

Strong type

$$\left|\alpha - \frac{p}{q}\right| < \frac{A}{q^{c+1}}$$

holds for infinitely many p/q for large A and finitely many p/q for small A.

Berry-Esseen type results

$X_1, X_2, \ldots$ i.i.d. integer valued nondegenerate random variables, α irrational, $S_n = \sum_{k=1}^n X_k$,

$$\Delta_n = \sup_x |P(\{S_n\alpha\} < x) - x|$$

Theorem. If $EX_1^2 < \infty$, S_n is unimodal and α is of strong type γ, then

$$\Delta_n = O(n^{-1/(2\gamma)}), \qquad \Delta_n = \Omega(n^{-1/(2\gamma)}).$$

$\gamma = 1$: badly approximable number α (bounded digits in continued fraction)

Theorem. Let $0 < \beta < 2$ and assume

$$P(|X_1| > t) \sim ct^{-\beta} \quad \text{and} \quad \lim_{x\to\infty} P(X_1 \geq x)/P(|X_1| \geq x) \quad \text{exists.}$$

If S_n is unimodal and α has strong type γ, then

$$\Delta_n = O(n^{-1/(\beta\gamma)}), \qquad \Delta_n = \Omega(n^{-1/(\beta\gamma)}).$$

Berry-Esseen problem for ordinary i.i.d. sums:

$$\sup_x \left| P\left(\frac{S_n}{n^{1/\beta}} < x\right) - G_\beta(x)\right| = O(n^{1-2/\beta})$$

A random version of Schmidt's lower bound

For any infinite sequence (x_k)

$$D_N(x_k) = \Omega\left(\frac{\log N}{N}\right)$$

and this bound is attained, e.g., for $x_k = \{k\alpha\}$ with badly approximable α

Theorem. For any nondegenerate i.i.d. sequence (X_n) and any irrational α

$$D_N(\{S_k\alpha\}) = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

This bound is attained if $P(|X_1| > x) \sim 1/\log x$

Critical behavior at $\gamma = 2$

Theorem. For smooth periodic f and $\gamma < 2$ we have

$$N^{-1/2}\sum_{k=1}^{N} f(S_k\alpha) \xrightarrow{d} N(0, \sigma^2)$$

and this fails for $\gamma > 2$.

$\gamma = 2$: $\quad \left|\alpha - \frac{p}{q}\right| < \frac{C}{q^3}$

$$\sum_{k=1}^{N} \frac{1}{k \|k\alpha\|^{1/2}} = \begin{cases} O(\log N) & \text{if } \gamma \leq 2, \\ O\left(N^{\gamma/2-1}\right) & \text{if } \gamma > 2 \end{cases}$$

Some classical examples for critical phenomena:

Lacunary trigonometric series (Erdős 1962) $\sum_{k=1}^{N} \sin n_k x, \quad \mathbf{n_k \gg e^{\sqrt{k}}}$

Gaussian processes (Dobrushin, Major, Taqqu 1979) $\sum_{k=1}^{N} f(\xi_k) \quad \mathbf{r_n \sim \frac{1}{n}}$

Critical discrepancy behavior

(i) If $1 \leq \gamma \leq 2$, then

$$D_N = O\left(\sqrt{\frac{\log\log N}{N}} \log N\right), \quad D_N = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

(ii) If $\gamma > 2$, then

$$D_N = O\left(\left(\frac{\log\log N}{N}\right)^{1/\gamma}\right), \quad D_N = \Omega\left(\frac{1}{N^{1/\gamma}}\right) \quad \text{a.s.}$$