Basic properties of three-dimensional continued fractions

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Recall that $GL_s(\mathbb{R})$, $s = 2, 3, \dots$ is the multiplicative group of nondegenerate $s \times s$ matrices with real entries, and let $GL_s(\mathbb{Z})$ be its discrete subgroup consisting of matrices with integer entries and determinant ± 1.

Any complete lattice in $\mathbb{R}^s$ can be written in the form

$$\Gamma(M) = \left\{ \gamma = m_1 \gamma^{(1)} + \cdots + m_s \gamma^{(s)} \mid m_1, \dots, m_s \in \mathbb{Z} \right\},$$

where $\gamma^{(1)}, \dots, \gamma^{(s)}$ are the basis nodes specified by the corresponding columns of a certain matrix $M \in GL_s(\mathbb{R})$.

The equality $\Gamma(M) = \Gamma(M')$ is true if and only if $M = M' \cdot S$, where $S \in GL_s(\mathbb{Z})$.

Denote by $\mathcal{L}_s(\mathbb{R})$ the set of all lattices.

Let G_s be the finite group acting on $GL_s(\mathbb{R})$ generated by transformations of the two following types:
(a) change of sign of a row or column;
(b) transposition of two rows or two columns.
Two matrices from $GL_s(\mathbb{R})$ are called *equivalent* if one of them is obtained from the other under a certain transformation from G_s.

The two-dimensional case ($s = 2$). A Voronoi basis

Let $\{\gamma^{(1)}, \gamma^{(2)}\}$ be a basis of the lattice Γ from $\mathcal{L}_s(\mathbb{R})$ such that Γ has no nonzero nodes $\gamma = (\gamma_1, \gamma_2)$ with

$$|\gamma_1| < \max\left\{\left|\gamma_1^{(1)}\right|, \left|\gamma_1^{(2)}\right|\right\}, \qquad |\gamma_2| < \max\left\{\left|\gamma_2^{(1)}\right|, \left|\gamma_2^{(2)}\right|\right\}.$$

This is true if and only if, for a certain transformation Φ from G_2, we have

$$\Phi \begin{pmatrix} \gamma_1^{(1)} & \gamma_1^{(2)} \\ \gamma_2^{(1)} & \gamma_2^{(2)} \end{pmatrix} = M = \begin{pmatrix} a_1 & -b_1 \\ a_2 & b_2 \end{pmatrix}, \qquad 0 \le b_1 \le a_1, \quad 0 \le a_2 \le b_2.$$

A basis $\{\gamma^{(1)}, \gamma^{(2)}\}$ of an arbitrary lattice Γ from $\mathcal{L}_2(\mathbb{R})$ is *a Voronoi basis* if its corresponding matrix is equivalent to the matrix M.

The classical continued fraction and 2D lattices

Let $[t_0; t_1, \ldots, t_i, \ldots]$ be the continued fraction representation of real α, where $t_0 \in \mathbb{Z}$ and $t_i \in \mathbb{Z}_+ \quad (i = 1, 2, \ldots)$ are called the partial quotients of the continued fraction and

$$p_i/q_i = [t_0; t_1, \ldots, t_i] \quad (i = 1, 2, \ldots) \quad \text{and} \quad p_0/q_0 = 1/0$$

are called the convergents of the continued fraction.
We assign to $\alpha \in (0, 1/2)$ the lattice

$$\Gamma_\alpha = \left\{ (\alpha n + m, n) = n(\alpha, 1) + m(1, 0) \;\middle|\; m, n \in \mathbb{Z} \right\}.$$

For this lattice, $\det(\Gamma_\alpha) = 1$, and, by the Lagrange Best Approximation Theorem, any Voronoi basis Γ_α consists of the pairs of nodes

$$\pm(\alpha q_i - p_i, q_i), \qquad \pm(\alpha q_{i+1} - p_{i+1}, q_{i+1}) \qquad (i = 0, 1, 2, \ldots).$$

Vahlen's theorem

In the theory of continued fractions these are well known inequalities

$$q_i|\alpha q_i - p_i| \leq 1, \qquad \min\{q_i|\alpha q_i - p_i|,\ q_{i+1}|\alpha q_{i+1} - p_{i+1}|\} \leq \frac{1}{2}.$$

The second one of them is usually called Vahlen's theorem.
Notice that for the matrices M

$$\frac{a_1 a_2 + b_1 b_2}{\det(M)} = \frac{a_1 a_2 + b_1 b_2}{a_1 b_2 + a_2 b_1} = 1 - \frac{(a_1 - b_1)(b_2 - a_2)}{a_1 b_2 + a_2 b_1} \leq 1.$$

Therefore, for any Voronoi basis, we have the inequality

$$\left|\gamma_1^{(1)} \cdot \gamma_2^{(1)}\right| + \left|\gamma_1^{(2)} \cdot \gamma_2^{(2)}\right| \leq \det(\Gamma) = \det(M),$$

which becomes an equality for $a_1 = b_1$ or $a_2 = b_2$. It follows that, for any node from a Voronoi basis,

$$|\gamma_1\gamma_2| \leq \det(\Gamma) \quad \text{and} \quad \min\left\{\left|\gamma_1^{(1)}\gamma_2^{(1)}\right|, \left|\gamma_1^{(2)}\gamma_2^{(2)}\right|\right\} \leq \frac{1}{2}\det(\Gamma).$$

The Markov-Hurwitz theorem

For three successive convergents it is known

Theorem (in the terms of continued fractions)

$\forall i = 1, 2, \ldots$

$$\min\left\{\left|\alpha - \frac{p_{i-1}}{q_{i-1}}\right| q_{i-1}^2, \left|\alpha - \frac{p_i}{q_i}\right| q_i^2, \left|\alpha - \frac{p_{i+1}}{q_{i+1}}\right| q_{i+1}^2\right\} \leq \frac{1}{\sqrt{5}}.$$

Theorem (for the nodes of a Voronoi basis)

If $\{\gamma^{(1)}, \gamma^{(2)}\}$ *is a Voronoi basis, then*

$$\frac{\min\left\{\left|\gamma_1^{(1)}\gamma_2^{(1)}\right|, \left|\gamma_1^{(2)}\gamma_2^{(2)}\right|, \left|\left(\gamma_1^{(1)} + \gamma_1^{(2)}\right)\left(\gamma_2^{(1)} + \gamma_2^{(2)}\right)\right|\right\}}{\det(\Gamma)} \leq \frac{1}{\sqrt{5}}.$$

Here the third element is defined by the sum of the nodes of a Voronoi basis.

Minkowski bases

A basis $\{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}\}$ of a lattice Γ from $\mathcal{L}_3(\mathbb{R})$ is called a Minkowski basis if the corresponding matrix is equivalent to a matrix of one of the following forms

$$\begin{pmatrix} a_1 & b_1 & -c_1 \\ -a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix}, \qquad \begin{pmatrix} a_1 & b_1 & c_1 \\ -a_2 & b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{pmatrix}$$

with nonnegative a_i, b_i, c_i such that
(i) $\max\{b_1, c_1\} \leq a_1$, $\max\{a_2, c_2\} \leq b_2$, $\max\{a_3, b_3\} \leq c_3$;
(ii) matrices of the first form satisfy at least one of the inequalities
$a_2 \leq c_2$, $b_3 \leq a_3$;
(iii) matrices of the second form satisfy the inequalities
$c_1 \leq b_1$, $b_2 \leq a_2 + c_2$.

Vahlen's theorem for Minkowski bases

Notice that Minkowski's convex body theorem immediately implies the estimate

$$|\gamma_1\gamma_2\gamma_3| \leq \det(\Gamma)$$

for any relative minimum $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ of a three-dimensional lattice Γ from $\mathcal{L}_3(\mathbb{R})$. In 1999 it was shown that for adjacent relative minima $\gamma^{(1)}$ and $\gamma^{(2)}$

$$\min\left\{\left|\gamma_1^{(1)}\gamma_2^{(1)}\gamma_3^{(1)}\right|, \left|\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}\right|\right\} \leq \frac{1}{2}\det(\Gamma).$$

For nodes $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$ that form a Minkowski basis in 2003 the estimate

$$\min\left\{\left|\gamma_1^{(1)}\gamma_2^{(1)}\gamma_3^{(1)}\right|, \left|\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}\right|, \left|\gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)}\right|\right\} \leq \frac{1}{3}\det(\Gamma)$$

was proved and in 2006 the following improvement was achieved:

$$\left|\gamma_1^{(1)}\gamma_2^{(1)}\gamma_3^{(1)}\right| + \left|\gamma_1^{(2)}\gamma_2^{(2)}\gamma_3^{(2)}\right| + \left|\gamma_1^{(3)}\gamma_2^{(3)}\gamma_3^{(3)}\right| \leq \det(\Gamma).$$

3D Markov-Hurwitz

It is know that (Davenport, 1938–43)

$$\min_{\gamma\in\Gamma\setminus\{0\}} \left|\gamma_1\gamma_2\gamma_3\right| \leq \frac{1}{7}\,\mathrm{det}(\Gamma).$$

This inequality is sharp. There is a unique lattice Γ such that

$$\min_{\gamma\in\Gamma\setminus\{0\}} \left|\gamma_1\gamma_2\gamma_3\right| = \frac{1}{7}\,\mathrm{det}(\Gamma).$$

This lattice is generated by columns of the matrix

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{pmatrix}$$

where

$$\{\alpha,\beta,\gamma\} = \left\{2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7}\right\}.$$

The constant $\frac{1}{7}$ is a first element of 3D analogue of Markov spectrum. Second element is $\frac{1}{9}$. It corresponds to a lattice Γ generated by columns of the matrix

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{pmatrix}$$

where

$$\{\alpha, \beta, \gamma\} = \left\{ 2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9} \right\}.$$

The begining of 3D Markov spectrum was calculated by Swinnerton–Dyer (1971). Only first 20 elements are known. The reciprocals are

$$7, 9, \sqrt{148}, \frac{63}{5}, 13, 14, \frac{351}{25}, \frac{189}{13}, \frac{133}{19}, \sqrt{229}, \frac{259}{17}, \ldots$$

From "global"to "local"

Cassels (An introduction to the geometry of numbers):
"It would be interesting if local methods could be successfully extended to problems in more than 2 dimensions, for example to problems relating to $x_1 \max\{x_2^2, x_3^2\}$, $x_1(x_2^2 + x_3^2)$, $x_1^2 + x_2^2 - x_3^2$ *or* $x_1x_2x_3$. *The difficulty is not to find the analogues of the* x_i *but to devise techniques to cope with their interrelations."*
Our problem is a part of this program, we want to replace infinite set $\Gamma\backslash\{0\}$ in formula

$$\min_{\gamma\in\Gamma\backslash\{0\}} |\gamma_1\gamma_2\gamma_3| \leq \frac{1}{7}\det\Gamma$$

by finite set of nodes.

Conjectures

1) Let $\Gamma = \langle \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \rangle$ where $\{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}\}$ is a Minkovski basis of the I type and

$$\Gamma_1 = \{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(1)} - \gamma^{(3)}, \gamma^{(1)} + \gamma^{(2)} - \gamma^{(3)}, \gamma^{(1)} + \gamma^{(2)}, \gamma^{(2)} + \gamma^{(3)}\}.$$

Then

$$\min_{\gamma \in R(\Gamma_1)} |\gamma_1 \gamma_2 \gamma_3| \leq \frac{1}{7} \det \Gamma$$

2) Let $\Gamma = \langle \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)} \rangle$ where $\{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}\}$ is a Minkovski basis of the II type and

$$\Gamma_2 = \{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(1)} + \gamma^{(2)}, \gamma^{(1)} - \gamma^{(2)} - \gamma^{(3)}, \gamma^{(1)} - \gamma^{(3)}, \gamma^{(2)} + \gamma^{(3)},$$

$2\gamma^{(1)} - \gamma^{(2)} - \gamma^{(3)}\}$.
Then

$$\min_{\gamma \in R(\Gamma_2)} |\gamma_1 \gamma_2 \gamma_3| \leq \frac{1}{9} \det \Gamma.$$

The main result

Theorem

1) If for all $\gamma \in \Gamma_1$ products $|\gamma_1\gamma_2\gamma_3|$ are equal to each other then Γ is generated by $2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7}$.
2) If for all $\gamma \in \Gamma_2$ products $|\gamma_1\gamma_2\gamma_3|$ are equal to each other then Γ is generated by $2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}$.