On Distances to Lattice Points in Knapsack Polyhedra

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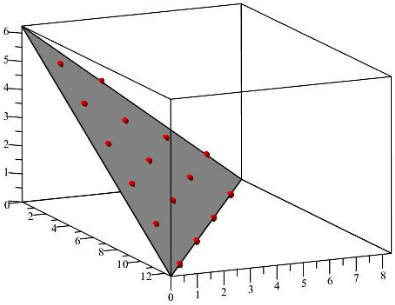
Statement of the problem

Given $\boldsymbol{a} \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, a knapsack polyhedron $P(\boldsymbol{a}, b)$ is defined as

$$P(\boldsymbol{a}, b) = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \boldsymbol{a} \cdot \boldsymbol{x} = b\}.$$

We are interested in the distance from a vertex of $P(\boldsymbol{a}, b)$ to the set of its lattice points.



(Lattice points in $P((2, 3, 4), 25)$)

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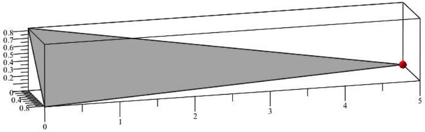


(Another example: Lattice points in $P((6,1,6),5)$)

We will estimate the (maximum) vertex distance

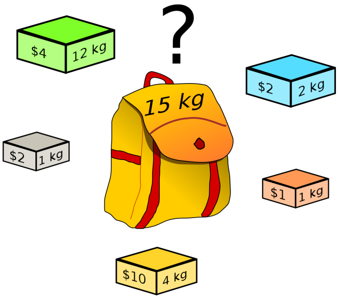
$$d(\boldsymbol{a}, b) = \begin{cases} \max_{\boldsymbol{v}} \min_{\boldsymbol{z} \in P(\boldsymbol{a},b)\cap\mathbb{Z}^n} \|\boldsymbol{v} - \boldsymbol{z}\|_\infty, & \text{if } P(\boldsymbol{a}, b) \cap \mathbb{Z}^n \neq \emptyset, \\ -\infty, & \text{otherwise}, \end{cases}$$

where the maximum is taken over all vertices $\boldsymbol{v}$ of the polyhedron $P(\boldsymbol{a}, b)$.

Link to Mathematical Optimisation

Given a cost vector $\boldsymbol{c} \in \mathbb{Q}^n$, $\boldsymbol{a} \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, the classical integer knapsack problem in Integer Programming is stated as

$$\max\{\boldsymbol{c} \cdot \boldsymbol{x} : \boldsymbol{x} \in P(\boldsymbol{a}, b) \cap \mathbb{Z}^n\}\,. \tag{1}$$



(Credits to Wikipedia)

The problem (1) is NP-hard (Lueker (1975)).

Link to Mathematical Optimisation

The linear programming relaxation of (1)

$$\max\{\boldsymbol{c} \cdot \boldsymbol{x} : \boldsymbol{x} \in P(\boldsymbol{a}, b)\} \tag{2}$$

can be solved in polynomial time (Khachiyan (1979)).

Let $IP(\boldsymbol{c}, \boldsymbol{a}, b)$ and $LP(\boldsymbol{c}, \boldsymbol{a}, b)$ denote the optimal values of (1) and (2).
We can bound the integrality gap

$$LP(\boldsymbol{c}, \boldsymbol{a}, b) - IP(\boldsymbol{c}, \boldsymbol{a}, b) \leq d(\boldsymbol{a}, b)\|\boldsymbol{c}\|_1 .$$

Bounds for the vertex distance

We will assume the following conditions:

$$\begin{array}{ll} (i) & \boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \,, n \geq 2 \,, a_i \neq 0 \,, i = 1, \ldots, n \,, \\ (ii) & \gcd(\boldsymbol{a}) := \gcd(a_1, \ldots, a_n) = 1 \,. \end{array} \tag{3}$$

Theorem 1

(i) *Let $\boldsymbol{a}$ satisfy (3) and $b \in \mathbb{Z}$. Then*

$$d(\boldsymbol{a}, b) \leq \|\boldsymbol{a}\|_\infty - 1 \,.$$

(ii) *For any positive integer k and any dimension n there exist $\boldsymbol{a}$ satisfying (3) with $\|\boldsymbol{a}\|_\infty = k$ and $b \in \mathbb{Z}$ such that*

$$d(\boldsymbol{a}, b) = \|\boldsymbol{a}\|_\infty - 1 \,.$$

Some known results

Results of Cook et al. (1986) imply

$$d(\boldsymbol{a}, b) \leq n\|\boldsymbol{a}\|_\infty.$$

Let $A = (a_{ij}) \in \mathbb{Z}^{m \times n}$, $\boldsymbol{b} \in \mathbb{Z}^m$. A recent result of Eisenbrand and Weismantel (2017) implies that to every vertex $\boldsymbol{v}$ of an integer feasible polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}$ there exists an integer point $\boldsymbol{z}$ in P, such that

$$\|\boldsymbol{v} - \boldsymbol{z}\|_1 \leq m(2m\|A\|_\infty + 1)^m.$$

Vertex distance in a randomised scenario

How large is the vertex distance of a "typical" knapsack polyhedron? Specifically, consider for $H \geq 1$ the set $Q(H)$ of $\boldsymbol{a} \in \mathbb{Z}^n$ that satisfy (3) and

$$\|\boldsymbol{a}\|_\infty \leq H\,.$$

For $\epsilon \in (0, 3/4)$ let

$$N_\epsilon(H, t) = \#\left\{\boldsymbol{a} \in Q(H) : \max_{b \in \mathbb{Z}} \frac{d(\boldsymbol{a}, b)}{\|\boldsymbol{a}\|_\infty^\epsilon} > t\right\}.$$

Theorem 2

Fix $n \geq 3$. For any $\epsilon \in (0, 3/4)$ we have

$$\frac{N_\epsilon(H,t)}{\#(Q(H))} \ll_n t^{-\alpha(\epsilon,n)} \text{ with } \alpha(\epsilon, n) = \frac{n-2}{(1-\epsilon)n}$$

over all $H \geq 1$ and $t > 0$.

Average vertex distance

Corollary 3

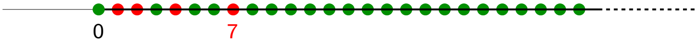
Fix $n \geq 3$*. For any* $\epsilon > 2/n$

$$\frac{1}{\#(Q(H))} \sum_{\boldsymbol{a} \in Q(H)} \max_{b \in \mathbb{Z}} \frac{d(\boldsymbol{a}, b)}{\|\boldsymbol{a}\|_\infty^\epsilon} \ll_n 1 \,.$$

Tools for positive $\boldsymbol{a}$

Let $\boldsymbol{a} \in \mathbb{Z}^n_{>0}$ with $\gcd(\boldsymbol{a}) = 1$. The Frobenius number $g(\boldsymbol{a})$ is the largest integer b that cannot be represented as a nonnegative integer linear combination of a_i-s, that is $P(\boldsymbol{a}, b) \cap \mathbb{Z}^n = \emptyset$.

For instance, let $\boldsymbol{a} = (3, 5)$. The representable (green dots) and non-representable (red dots) integers:



$g(\boldsymbol{a}) = 7$.

Covering number

Let $K \subset \mathbb{R}^n$ be a convex body, Λ a full-dimensional lattice and $S \in \{\mathbb{R}^n, \mathbb{Z}^n\}$.

We define the covering number

$$\mu(K, \Lambda, S) = \min\{\mu \ : \ S \subset \mu K + \Lambda\}.$$

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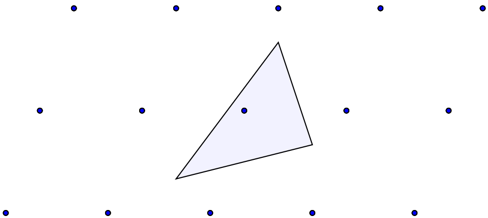

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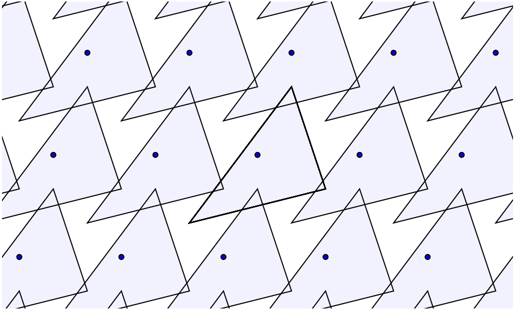

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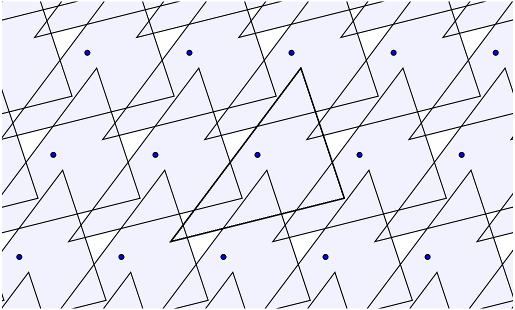

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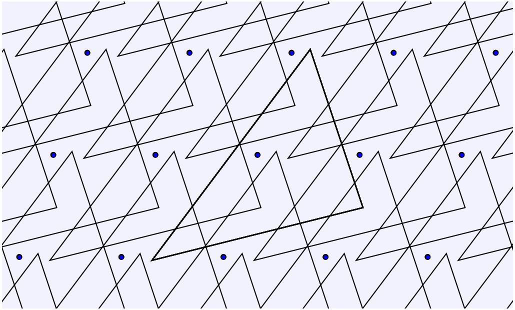

Results of Ravi Kannan

For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ with $\gcd(\boldsymbol{a}) = 1$, let

$$T_{\boldsymbol{a}} = \left\{ \boldsymbol{x} \in \mathbb{R}^{n-1}_{\geq 0} \; : \; \sum_{i=1}^{n-1} a_i x_i \leq 1 \right\}$$

and

$$\Lambda_{\boldsymbol{a}} = \left\{ \boldsymbol{x} \in \mathbb{Z}^{n-1} \; : \; \sum_{i=1}^{n-1} a_i x_i \equiv 0 \bmod a_n \right\}.$$

Kannan (1992):

$$\mu(T_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}, \mathbb{R}^{n-1}) = g(\boldsymbol{a}) + a_1 + \ldots + a_n$$

and

$$\mu(T_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}, \mathbb{Z}^{n-1}) = g(\boldsymbol{a}) + a_n.$$

Idea of the proof of Theorem 1 (i) for positive $\boldsymbol{a}$

Let $\boldsymbol{a} \in \mathbb{Z}^n_{>0}$ satisfy (3) and $b \in \mathbb{Z}$. Key inequality for positive case:

Lemma 4

$$d(\boldsymbol{a}, b) \leq \frac{\mu(T_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}, \mathbb{Z}^{n-1})}{\min_i a_i} \leq \frac{g(\boldsymbol{a}) + \|\boldsymbol{a}\|_\infty}{\min_i a_i} .$$

Idea of the proof of Theorem 2 for positive $\boldsymbol{a}$

For convenience, we will work with the quantity

$$f(\boldsymbol{a}) = g(\boldsymbol{a}) + a_1 + \cdots + a_n .$$

Let $s(\boldsymbol{a}) = a_{n-1} a_n^{1/(n-1)}$ and

$$R = \{\boldsymbol{a} \in \mathbb{Z}^n : 0 < a_1 \leq \cdots \leq a_n\}$$

Idea of the proof of Theorem 2 for positive $\boldsymbol{a}$

The next key lemma is a special case of a result of Strömbergsson (2012) on asymptotic distribution of Frobenius numbers (for $n = 3$ can be derived from results of Shur, Sinai and Ustinov (2009)).

Lemma 5

$$\#\left\{\boldsymbol{a} \in Q(H) \cap R : \frac{f(\boldsymbol{a})}{s(\boldsymbol{a})} > r\right\} \ll_n \frac{1}{r^{n-1}} \#(Q(H)), \tag{4}$$

over all $r > 0$ and $H \geq 1$.

With some technical work we move from $s(\boldsymbol{a})$ to $\|\boldsymbol{a}\|_\infty^\epsilon$ and then apply Lemma 4.

Thank you for listening!