More on Geometric Bijections and Reversal Systems

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Outline

Recall from Spencer's talk: $\text{Jac}(M) \circlearrowright \mathcal{G}(M) \approx \chi(M) \leftrightarrow \mathcal{B}(M)$.

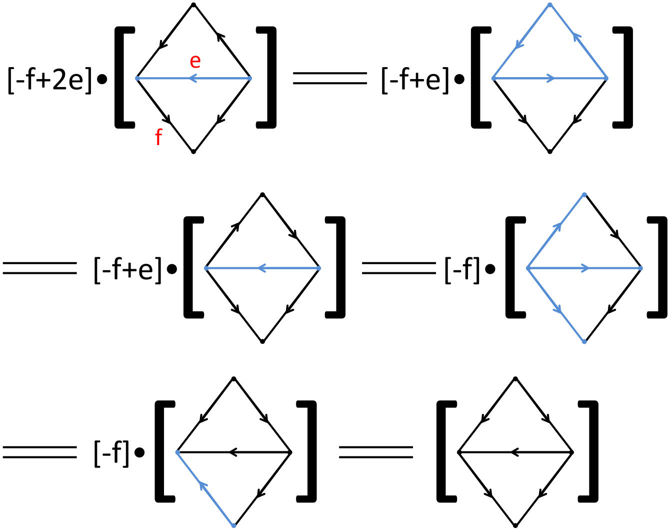
- $\text{Jac}(M)$: Jacobian.
- $\mathcal{G}(M)$: Circuit-cocircuit reversal system.
- $\chi(M)$: Circuit-cocircuit minimal orientations.
- $\mathcal{B}(M)$: Bases.

1. A *group action-tiling duality* for regular matroids (with examples).
2. $\mathcal{G}(M) \not\approx \chi(M)$ for non-regular oriented matroids.
   (Joint work with Emeric Gioan)
3. Geometric bijections $\chi(M) \leftrightarrow \mathcal{B}(M)$ for general oriented matroids.
   (Joint work with Spencer Backman and Francisco Santos)

More Details of $\mathrm{Jac}(M) \circlearrowright \mathcal{G}(M)$

$\mathrm{Jac}(M) \cong \frac{C_1(M)}{B_1(M) \oplus Z_1(M)}$.

$C_1(M) = \mathbb{Z}^E$, $B_1(M)$: cocircuit (bond) lattice, $Z_1(M)$: circuit (flow) lattice.



Group Actions from Geometric Bijections

$$\mathrm{Jac}(M) \circlearrowright \mathcal{G}(M) \leftrightarrow \mathcal{B}(M)$$

Observation

Any (geometric) bijection between $\mathcal{G}(M)$ *and* $\mathcal{B}(M)$ *induces a group action on* $\mathcal{B}(M)$ *by* $\mathrm{Jac}(M)$.

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Observation

Different bijections may lead to isomorphic group actions.

Bernardi Process of Plane Graphs

Olivier Bernardi's process (08): Fix a starting edge (v, f) in a *plane* graph.

- For every spanning tree T, starting with (v, f), walk along edges in T.
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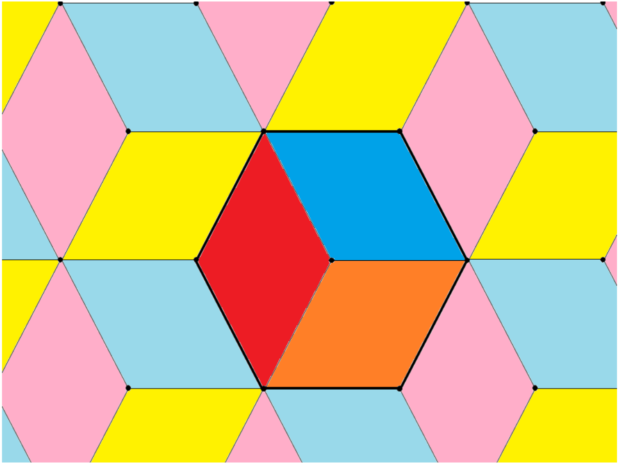
Theorem (Baker–Wang 2017, Chan–Church–Grochow 2015)

All Bernardi bijections (and rotor-routings*) of a plane graph induce isomorphic group actions.*

Tiling by Zonotopes

Theorem (Shephard 1974, McMullen 1975)

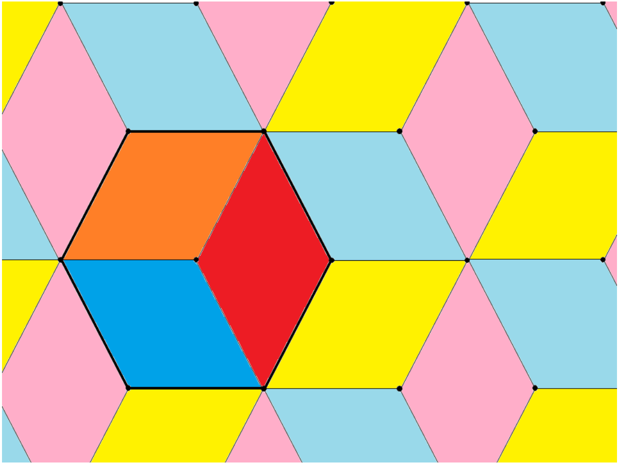
The zonotope of a matroid tiles the space iff the matroid is regular.



Tiling by Zonotopes

Observation

Many zonotopal tilings lead to the same tiling pattern.



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1. $\mathrm{Jac}(M) \cong \frac{C_1(M)}{B_1(M) \oplus Z_1(M)} \cong \frac{C_1(M^*)}{Z_1(M^*) \oplus B_1(M^*)} \cong \mathrm{Jac}(M^*)$.
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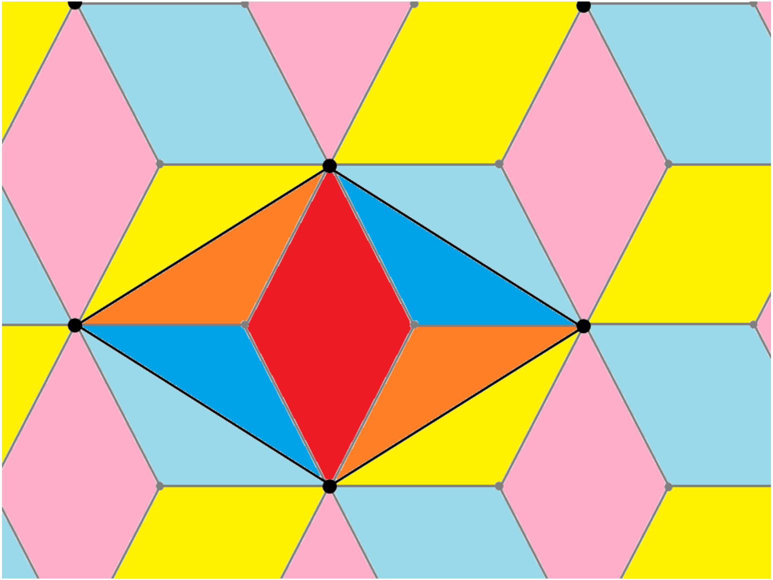
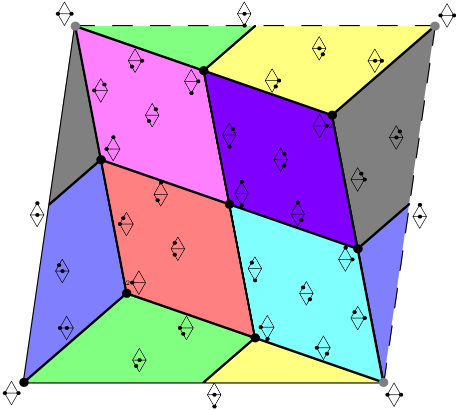
Punchline: The dual tilings of Bernardi processes were introduced in tropical geometry before.

Tiling by Jac(Γ)



ABKS Decomposition

An–Baker–Kuperberg–Shokrieh (2014): Construct a *canonical* decomposition of the *tropical Jacobian* Jac(Γ) of the tropical version of G.



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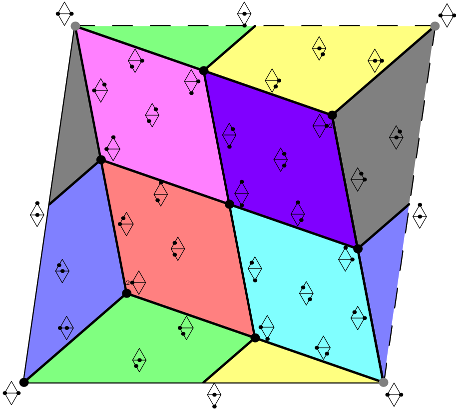

Proposition (Y. 2017+)

The ABKS decomposition of G^* is the dual of the Bernardi action of G.

Reversal Systems of Non-regular Matroids

Theorem (Gioan–Y. 2017+ (Converse of Gioan 2008))

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Corollary

It suffices to show that there exist equivalent CCMOs.

Non-regular Case (cont.)

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2. Choose carefully a reference ordering of elements, and a CCMO.
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Question

Does there exist $K > 1$ such that $|\mathcal{B}(M)| \geq K \cdot |\mathcal{G}(M)|$ for every non-regular M? More generally, how does the structure of M affect the inequality?

Circuit-cocircuit Minimal Orientations Revisited

Definition

Fix a generic single-element lifting $\widetilde{M}$ with signature σ, and a generic single-element extension M' with signature σ^*.
An orientation $\mathcal{O}$ is *(σ, σ^*)-compatible* if $(\mathcal{O}\ -)$ is acyclic in $\widetilde{M}$ and totally cyclic in M'.

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Example

CCMOs are compatible orientations with respect to some lexicographic lifting and extension.

Intuition: A generic circuit signature $\sigma : \mathcal{C}(M) \to \{+, -\}$ specifies a reference orientation for every circuit of M, i.e., the one with $\sigma(C) = +$. Same for σ^*.

Geometric Bijections of General Oriented Matroids

Theorem (Backman-Santos-Y. 2017+)

For any σ and σ^*, the number of (σ, σ^*)-compatible orientations equals the number of bases. Moreover, geometric bijection provides an explicit bijection.

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Proof Idea: The bijectivity of a geometric bijection is essentially the existence and uniqueness of optima of a family of bounded, feasible, generic *oriented matroid programs*.

Merci!