On the Generalization of the Matroid Parity Problem

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Marseille, 2018

COLLOQUE INTERNATIONAL DU CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

SUR LA THEORIE DES GRAPHES ET LA COMBINATOIRE

CENTRE INTERNATIONAL DE RENCONTRES MATHEMATIQUES

Marseille - Luminy

15 - 19 juin 1981

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Matroids

The pair (E, F) is a **matroid** (H. Whitney, 1935) if E is a finite set and F is a subset of 2^E so that

(F1) $\emptyset \; \varepsilon \; \mathrm{F}$
(F2) $X \; \varepsilon \; \mathrm{F}$ implies that $Y \; \varepsilon \; \mathrm{F}$ for every subset Y of X
(F3) If $X, Y \; \varepsilon \; \mathrm{F}$ and $|X| > |Y|$ then $\exists$ an element x in $X—Y$ so that $Y \cup \{x\} \; \varepsilon \; \mathrm{F}$

The rank function of a matroid

The elements of F are the **independent sets** of the matroid. For any subset *X* of *E* let *r(X)* denote the size of the maximal independent subsets of *X*. This function (the **rank function** of the matroid) can be characterized by the following set of properties:

(R1) $r(\emptyset) = 0$
(R2) If X is a subset of Y then $r(X) \leq r(Y)$
(R3) $r(X) \leq |X|$ for every subset X
(R4) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ (submodularity)

The k-matroid intersection problem

Input: k matroids $F_1, F_2, \ldots, F_k$ on the same underlying set E and an integer t

Question: Does there exist a subset X of E with $X \varepsilon F_1 \cap F_2 \cap \ldots \cap F_k$ and $|X| \geq t$?

We suppose that the matroids are given by **independence oracles,** that is, the answer of a question like „Does X belong to F_i ?" takes one step in an algorithm.

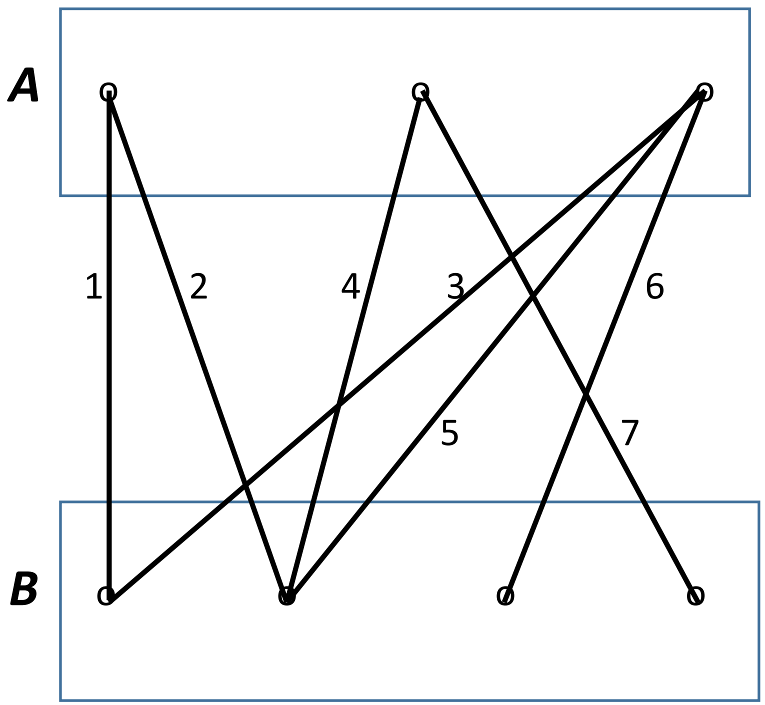
This problem is polynomially solvable for $k = 1$ (with the **greedy algorithm**) and for $k = 2$ (with the **matroid intersection algorithm** of J. Edmonds, 1965) but it is **NP**-hard for $k \geq 3$.

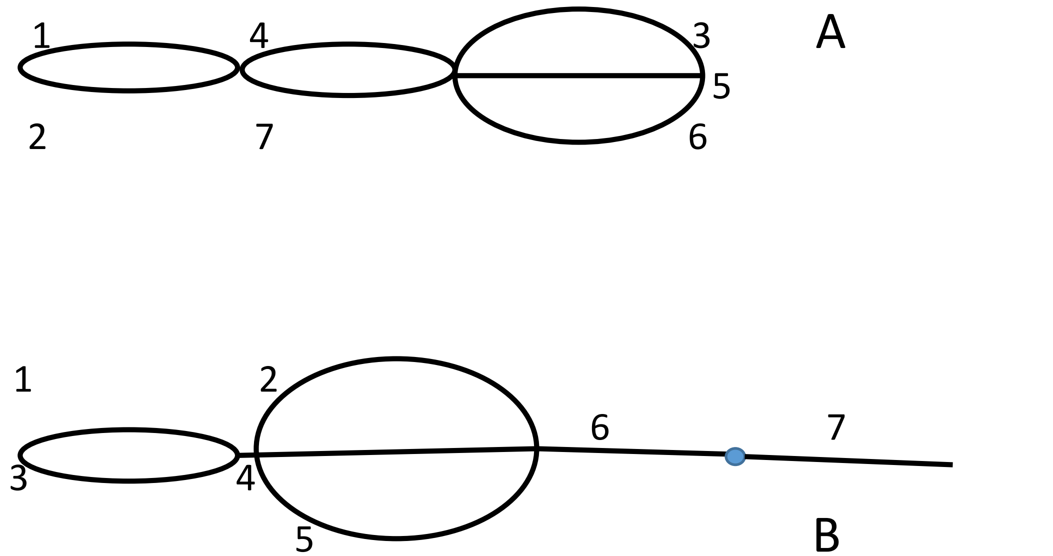
Bipartite matching as 2-matroid intersection

Let *G(A, B; E)* be a bipartite graph. We define two matroids A, B on *E* so that a subset *X* of *E* is independent in A or in B if and only if no two edges of *X* share a common vertex in *A* or in *B*, respectively.

G has a matching of size *t* if and only if A and B have a common independent set of cardinality *t.*

An example:





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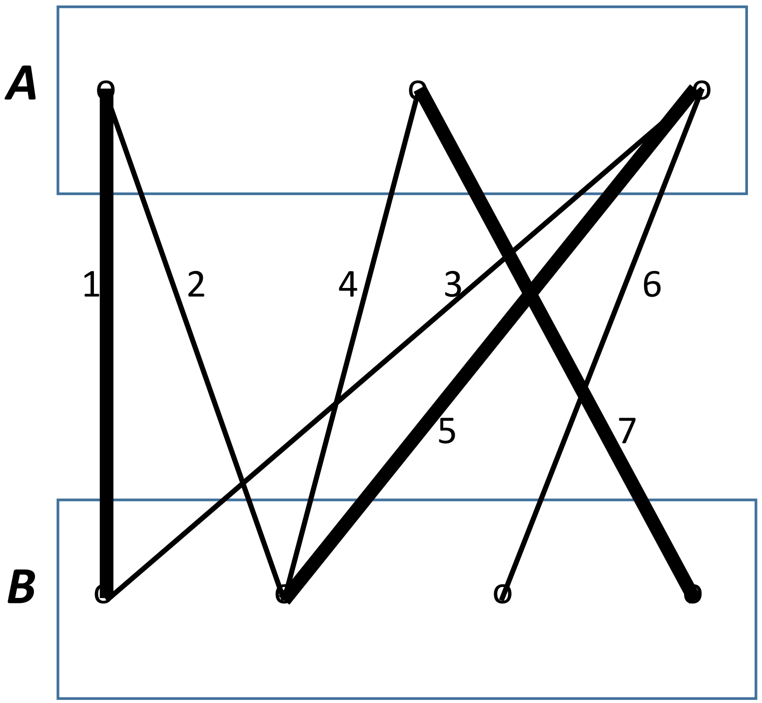

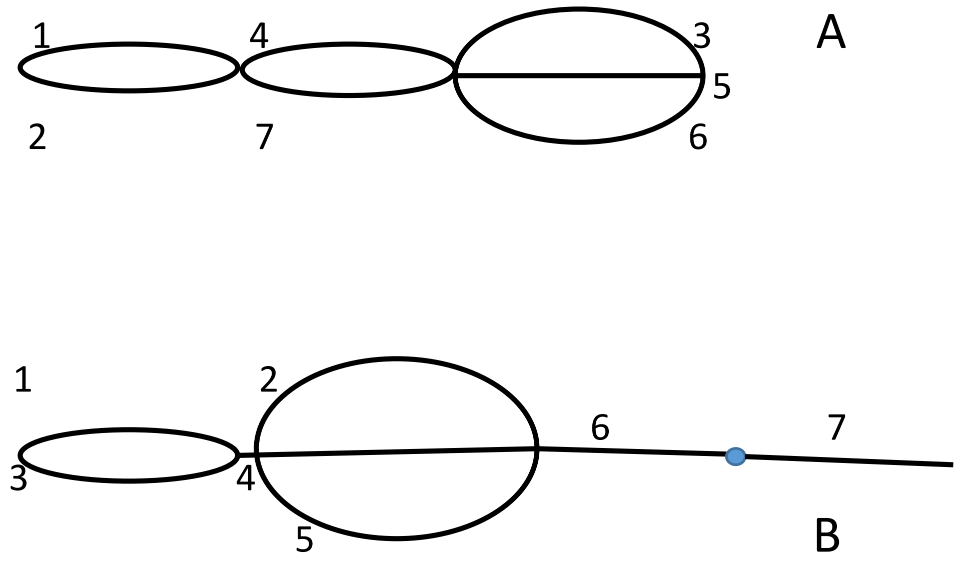

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G has a matching of size *t* if and only if A and B have a common independent set of cardinality *t.*

The existence of a matching of given size **in case of a non-bipartite graph** cannot be formulated as a 2-matroid intersection problem.

The polymatroid rank function

Recall that the rank function $r(X)$ of a matroid can be characterized by the following set of properties:

(R1) $r(\emptyset) = 0$
(R2) If X is a subset of Y then $r(X) \leq r(Y)$
(R3) $r(X) \leq |X|$ for every subset X
(R4) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ (submodularity)

If (R3) is replaced by

(R3') $r(X) \leq k \cdot |X|$ for every subset X

then we obtain the concept of *k*-**polymatroid rank function**

Two examples for polymatroid rank functions

Example 1   Let F_1, F_2, …, F_k be k matroids on the same underlying set E, with respective rank functions $r_1, r_2, \ldots, r_k$. Then $f_1(X) = r_1(X) + r_2(X) + \ldots + r_k(X)$ is a k-polymatroid rank function.

Example 2   Let $G(V, E)$ be a graph and X be a subset of E. Let $f_2(X)$ denote the number of vertices covered by X. Then $f_2(X)$ is a 2-polymatroid rank function.

k-polymatroid matching

Let $f(X)$ be a k-polymatroid rank function on a set E. A subset X of E is a k-**matching** if $f(X) = k \cdot |X|$

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In this example X is a k-matching if and only if $X \varepsilon F_i$ for every i.

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In this example X is a k-matching if and only if $X \varepsilon F_i$ for every i.

Example 2 Let $G(V, E)$ be a graph and X be a subset of E. Let $f_2(X)$ denote the number of vertices covered by X. Then $f_2(X)$ is a 2-polymatroid rank function.

In this example X is a 2-matching if and only if X is a matching in the graph G.

The k-polymatroid matching problem

Let $f(X)$ be a k-polymatroid rank function on a set E. A subset X of E is a k-**matching** if $f(X) = k \cdot |X|$

Input: A k-polymatroid rank function $f(X)$ on a set E, and an integer t
Question: Does there exist a k-matching of size t ?

We suppose that the k-polymatroid rank function is given by an **oracle,** that is, the answer of a question like „What is the value of f for a given subset?" takes one step in an algorithm.

This problem is polynomially solvable for $k = 1$ (with the greedy algorithm) and is **NP**-hard for $k \geq 3$ (since it contains the k-matroid intersection problem as a special case).

The matroid parity problem

The classical formulation:

Input: A matroid F on the underlying set $E = \{x_1, y_1, x_2, y_2, \dots, x_t, y_t\}$ and an integer p

Question: Does there exist a subset X of E with $X \varepsilon$ F and $|X| \geq p$, satisfying $|X \cap \{x_i, y_i\}| \neq 1$ for every i ?

(For each pair, either take both elements or none of them.)

The straightforward generalization (also known as the matroid *k*-parity problem)

Let T_1, T_2, … , T_t be disjoint *k*-element sets and let their union be denoted by *E*.

Input: A matroid F on the underlying set *E,* and a number *p*

Question: Does there exist a subset *X* of *E* with *X* ε F and with cardinality ≥ *p,* which intersects each T_i in either no or *k* elements?

(For each *k*-tuple, either take all the elements or none of them.)

The k-polymatroid matching problem

Let $f(X)$ be a k-polymatroid rank function on a set E. A subset X of E is a k-**matching** if $f(X) = k \cdot |X|$

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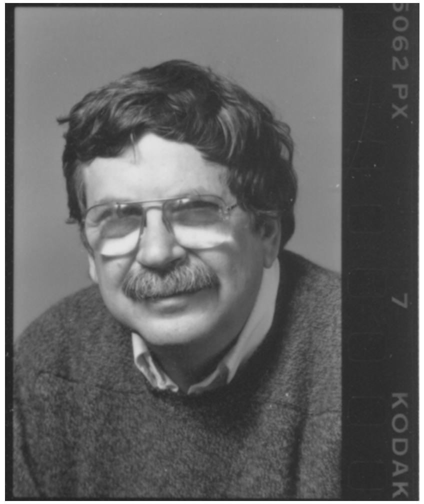
The case *k=2*

E. Lawler (1976) conjectured that the case *k=2* is polynomially solvable:



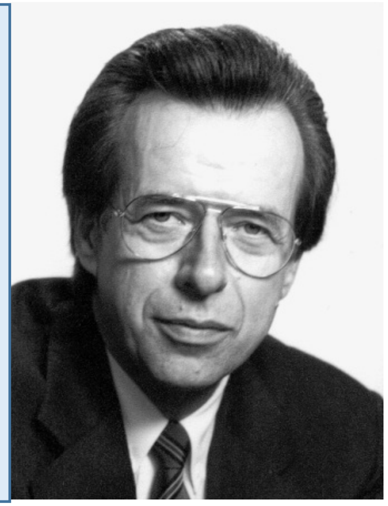
Matroid parity problem

Non-bipartite matching

2-matroid intersection problem

Bipartite matching

... but the truth is ...

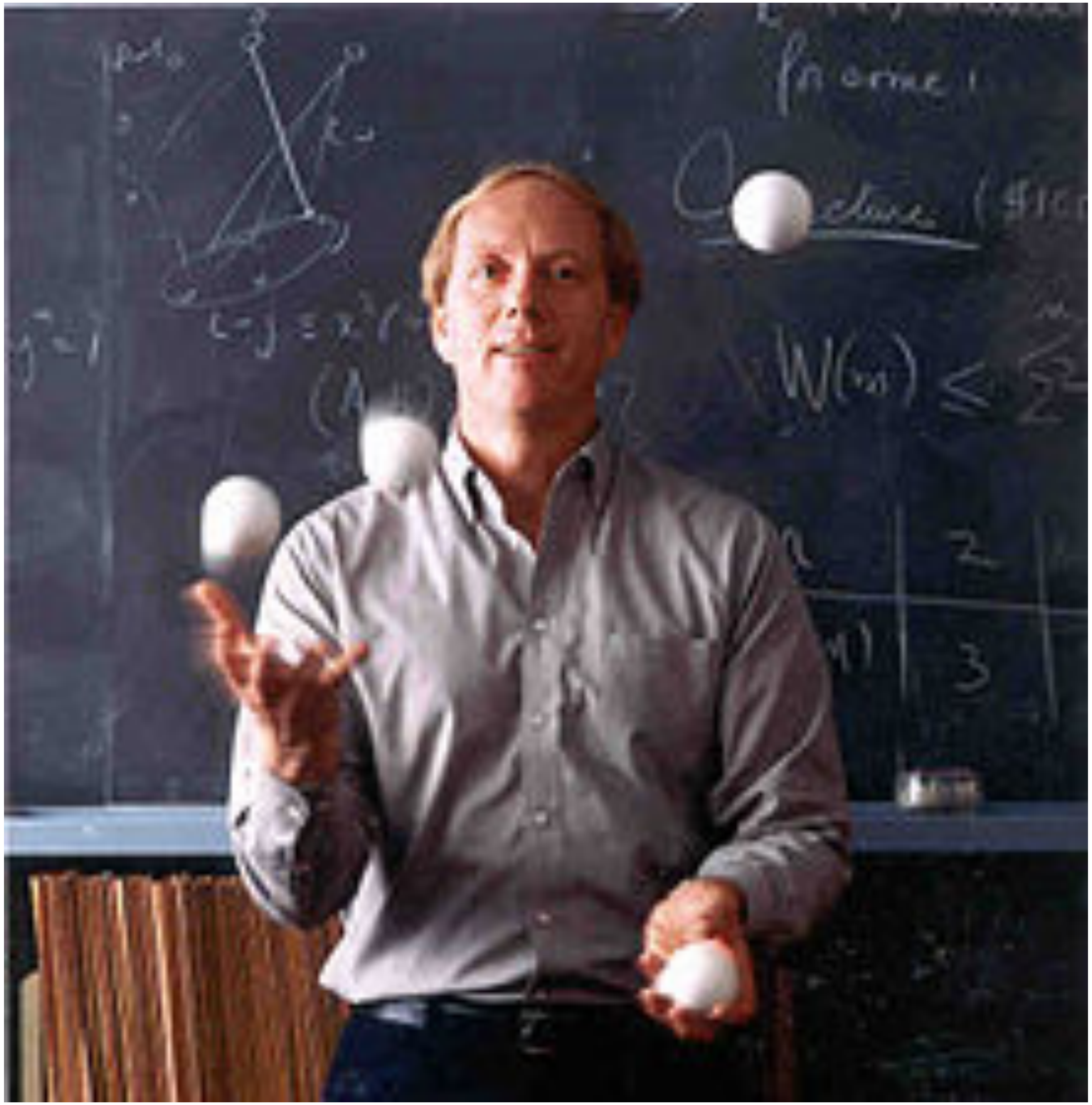


Bad news:
The matroid parity problem cannot be solved in polynomial time (L. Lovász, 1980 and P. M. Jensen – B. Korte, 1982)

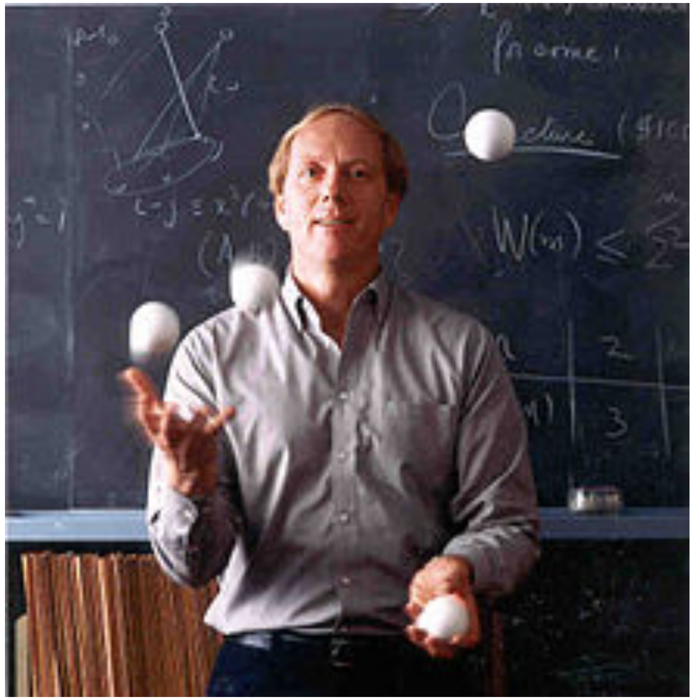
Good news:
The most important special case (which is required for the engineering applications) is polynomially solvable (L. Lovász, 1980)

Ronald Graham:

”An ideal math talk should have one proof and one joke and they should not be the same”.



Sketch proof of the negative result:



Let $|E| = 2t$ and let the function f be defined as follows:

If $|X| < t$ then let $f(X) = 2 \cdot |X|$

If $|X| > t$ then let $f(X) = 2t+1$

If $|X| = t$ then choose the value of $f(X)$ **randomly** from the set $\{2t-1, 2t\}$

All these functions are submodular, hence they are 2-polymatroid rank functions.

One needs exponential time to discover if there exists a t-element subset X satisfying $f(X) = 2t$.

A polynomially solvable special case

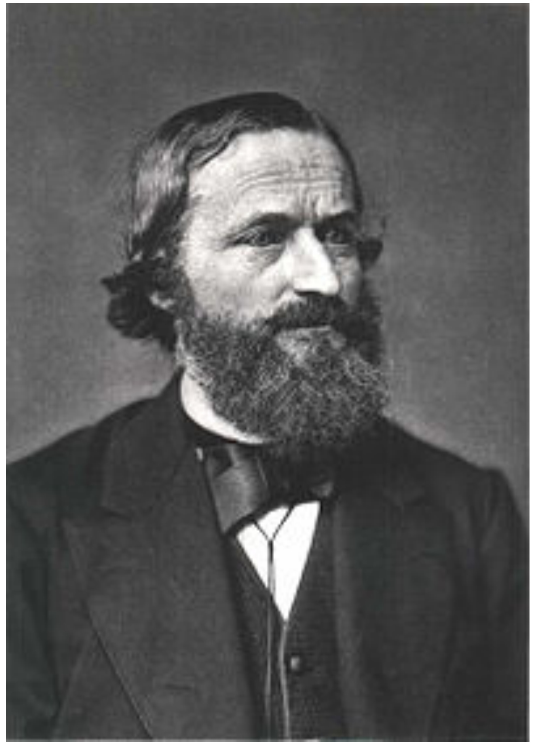
Let $S = \{x_1, y_1, x_2, y_2, \dots, x_t, y_t\}$ and let M be a matroid on *S*, represented by a matrix *M* over the field of the rationals. Let *E* be the set $\{1, 2, \dots, t\}$ of the subscripts and for any subset *X* of *E* let $f(X) = r(\cup\{x_i, y_i\} \mid i \varepsilon X)$, where *r* is the rank function of the matroid M. This *f* is clearly a 2-polymatroid rank function.

If a 2-polymatroid rank function *f* arises in this way (and if the matrix *M* is explicitly given) then the 2-polymatroid matching problem is polynomially solvable (L. Lovász, 1980).

Application #1   Electric network theory

Suppose at first that an electric network consists of voltage and current sources and positive resistors. Let *G* denote the graph of the interconnection of these elements.

The signed sum of the **voltages** along any **circuit** of *G* is zero and the signed sum of the **currents** along any **cut set** of *G* is zero (Kirchhoff, 1847).



Hence a necessary (and, in fact, also sufficient) condition of the unique solvability of the network is that the subgraph, formed by the voltage sources must be circuit-free and the subgraph formed by the current sources must be cut set free.

This is equivalent to the existence of a **normal tree:** this tree contains every voltage source edge and none of the current source edges.

Application #1 Electric network theory

Suppose that the electric network contains **ideal transformers** as well. These are 2-ports, represented by two edges *a, b* and described by the two equations $u_b = ku_a$ and $i_a = -ki_b$.

In this case the normal tree should contain every voltage source edge, none of the current source edges and exactly one edge from the pair $\{a, b\}$ for each transformer.

The existence of such a tree can be decided in polynomial time, using the 2-matroid intersection algorithm.

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In this case the normal tree should contain every voltage source edge, none of the current source edges and exactly one edge from the pair {*a, b*} for each transformer.

The existence of such a tree can be decided in polynomial time, using the 2-matroid intersection algorithm.

Suppose that the electric network contains **gyrators** as well. These are 2-ports, represented by two edges *a, b* and described by the two equations $u_b = Ri_a$ and $u_a = -Ri_b$.

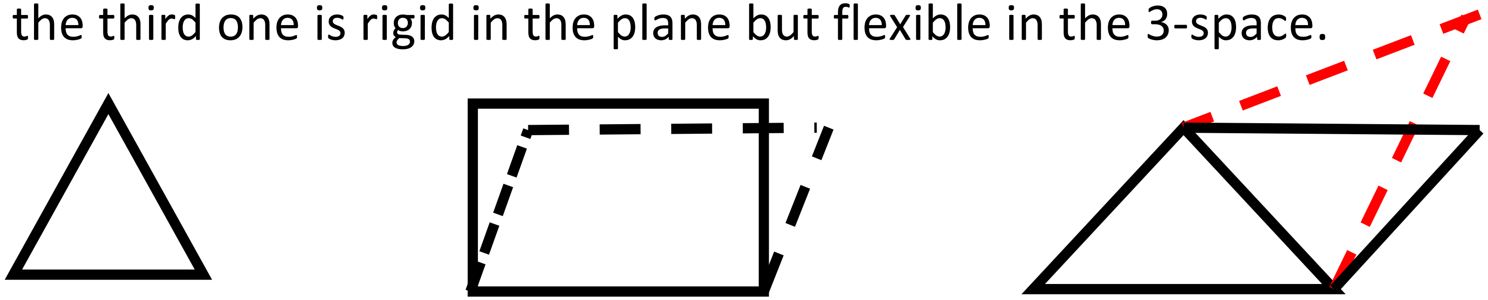
In this case the normal tree should contain every voltage source edge, none of the current source edges and either none of the edges or both edges from the pair {*a, b*} for each gyrator.

The existence of such a tree can be decided in polynomial time, using the 2-polymatroid matching algorithm of Lovász.

Application #2 Bar and joint frameworks

A framework consists of rigid bars, joined by rotatable joints.

It is intuitively clear that the first framework is rigid, the second one is flexible, while the third one is rigid in the plane but flexible in the 3-space.



What is the formal definition of rigidity?

The straightforward generalization (also known as the matroid *k*-parity problem)

Let T_1, T_2, … , T_t be disjoint *k*-element sets and let their union be denoted by *E*.

Input: A matroid F on the underlying set *E*, and a number *p*

Question: Does there exist a subset *X* of *E* with *X* ε F and with cardinality ≥ *p*, which intersects each T_i in either no or *k* elements?

(For each *k*-tuple, either take all the elements or none of them.)

Towards a generalization of the matroid *k*-parity problem

Let $T_1, T_2, \ldots, T_t$ be disjoint *k*-element sets and let their union be denoted by *E*. Let A be a non-empty subset of {0, 1, 2, …, *k*}.

Input: A matroid F with underlying set is *E*, the set A , and a number *p*
Question: Does there exist a subset *X* of *E* with $X \varepsilon$ F and with cardinality *p* , so that $|X \cap T_i|$ ε A holds for each subscript?

This problem reduces to the matroid *k*-parity problem if A = {0, *k*}.

The generalized matroid k-parity problem

Let $T_1, T_2, \ldots, T_t$ be disjoint k-element sets and let their union be denoted by E. Let A be a non-empty subset of $\{0, 1, 2, \ldots, k\}$.

For an integer $0 \leq c \leq t$ a subset X of E is called
($\geq c$)–**legal** if $|X \cap T_i| \in$ A holds for at least c subscripts

and it is called
c–**legal** if $|X \cap T_i| \in$ A holds for exactly c subscripts

Input: A matroid F with underlying set is E, A (and possibly a number p)
Question: Does there exist a ($\geq c$)–legal or a c–legal subset X of E with $X \in$ F (and possibly with given cardinality p)?

These problems reduce to the matroid parity problem if k=2, A = {0, 2} and $c = t$.

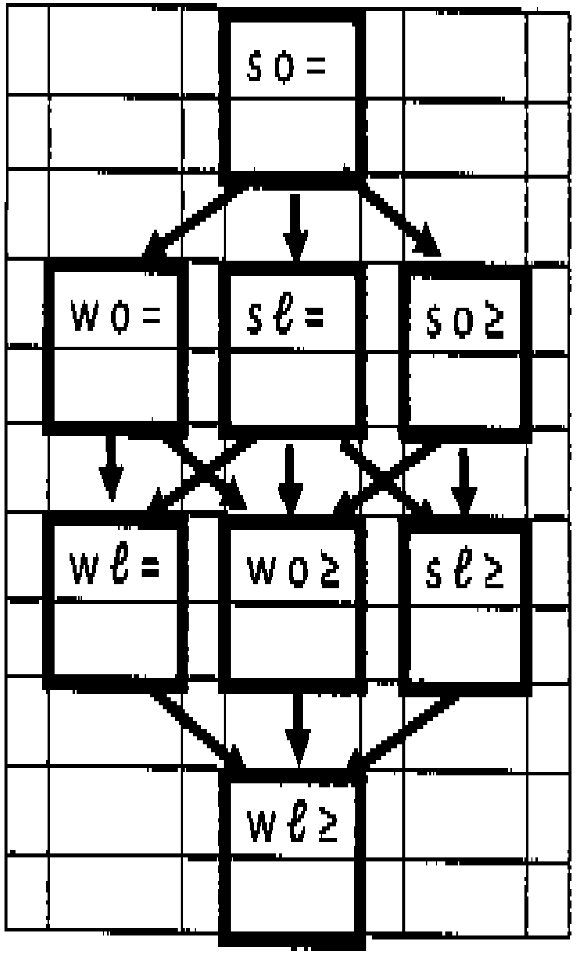
Some remarks

1. We speak about the *weak version* if there is no cardinality constraint, and about the *strong version* otherwise.

2. For a given A the complexity of the ($\geq c$)–legal version is at most t times that of the c–legal one but it is possible that only the former one is polynomial.

3. In view of the result of (Lovász, 1980) the complexity depends on the way how the matroid F is given (explicit linear representation over the field of the rationals or an independence oracle only).

Hence we have $2^3 = 8$ variations of this problem.



The first character:

- s for the strong version,
- w for the weak version

The second character:

- o if the matroid is given by an independence oracle,
- ℓ if it is given by a linear representation

The third character:

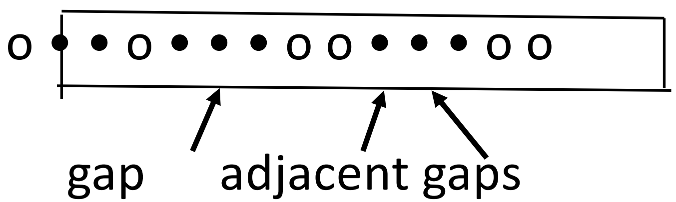
- = for the *c*-legal version,
- ≥ for the (≥*c*)-legal version

Our results (A. Recski, 1983 and joint work with Jácint Szabó, 2006)

If the matroid is given by linear representation then we have only partial results for the strong version but we could answer the other six cases.

Let B = {0, 1, 2, …, k} – A and let α and β denote the smallest elements of A and B, respectively.

A number k is a **gap** if k does not belong to A but there are other numbers $a<k<b$ so that a, b ε A .



The „weak version” of the problem

	F is given by an independence oracle	F is a represented linear matroid
($\geq c$)–legal sets	The problem is in **P** if and only if $\alpha \leq 1$	The problem is in **P** if $\alpha \leq 2$ and it is **NP**-complete if $\alpha \geq 3$

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c–legal sets	The problem is in **P** if and only if max $(\alpha,\beta) = 1$	The problem is in **P** if max $(\alpha,\beta) \leq 2$ and it is **NP**-complete if max $(\alpha,\beta) \geq 3$

The „strong version" of the problem

	F is given by an independence oracle	F is a represented linear matroid
($\geq c$)–legal sets	The problem is in **P** if and only if $\{1, 2, \ldots, k - 1\}$ is a subset of A	The problem is **NP**-complete unless if A contains no adjacent gaps and intersects both of the sets $\{0, 1, 2\}$ and $\{k - 2, k - 1, k\}$

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	F is given by an independence oracle	F is a represented linear matroid
$(\geq c)$–legal sets	The problem is in **P** if and only if $\{1, 2, \ldots, k-1\}$ is a subset of A	The problem is **NP**-complete unless if A contains no adjacent gaps and intersects both of the sets $\{0, 1, 2\}$ and $\{k-2, k-1, k\}$
c–legal sets	The problem is in **P** if and only if $k = 1$ (that is, never in **P** if $k > 1$)	The problem is **NP**-complete unless if A and B contain no adjacent gaps and they intersect both of the sets $\{0, 1, 2\}$ and $\{k-2, k-1, k\}$

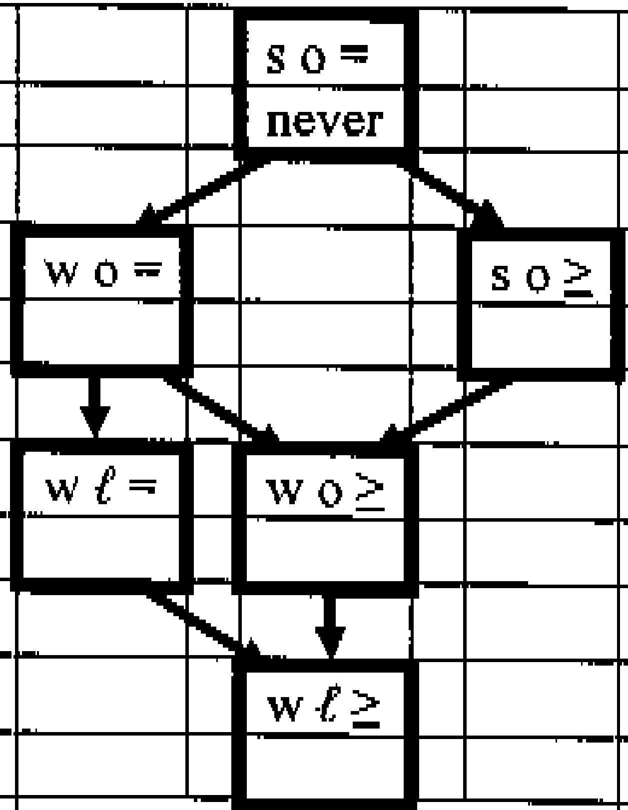


Figure 1
Subcases, solvable in polynomial time

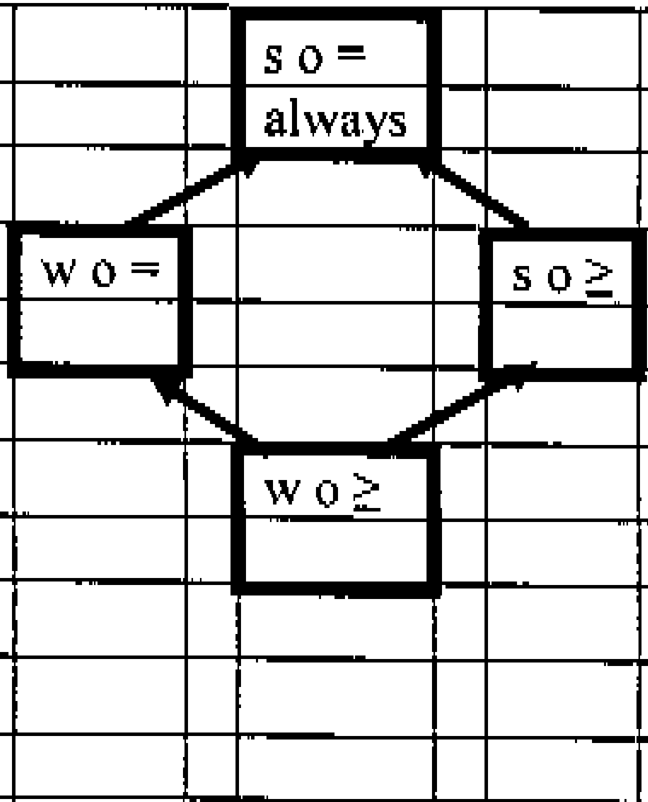


Figure 2

Non-polynomial subcases

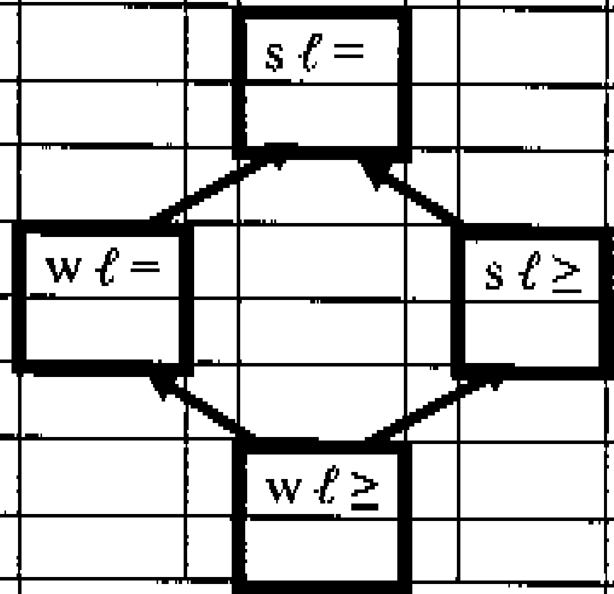
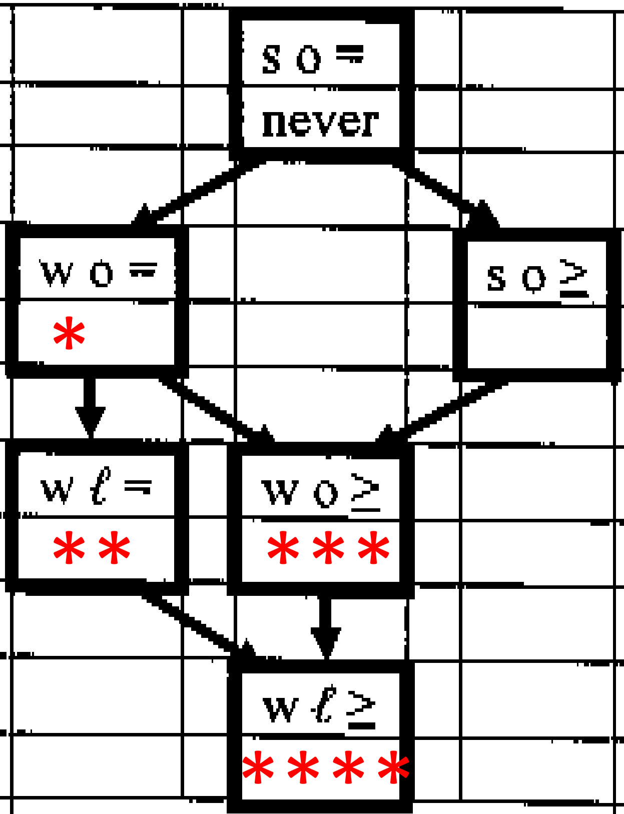


Figure 3

NP-complete subcases



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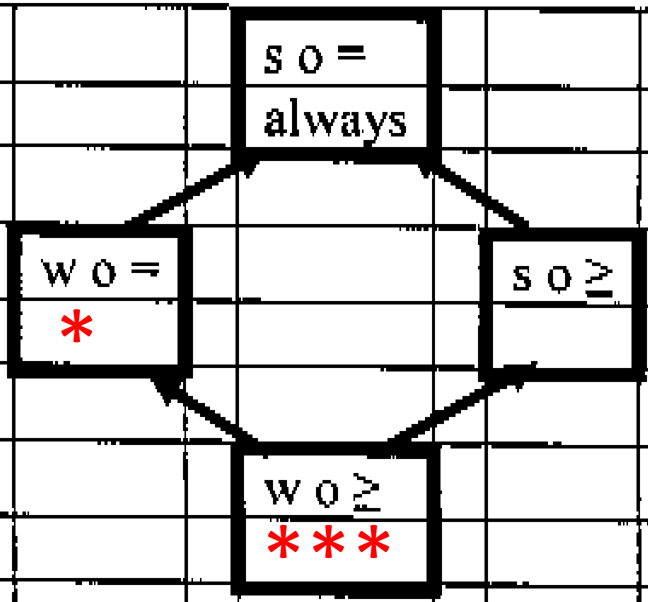

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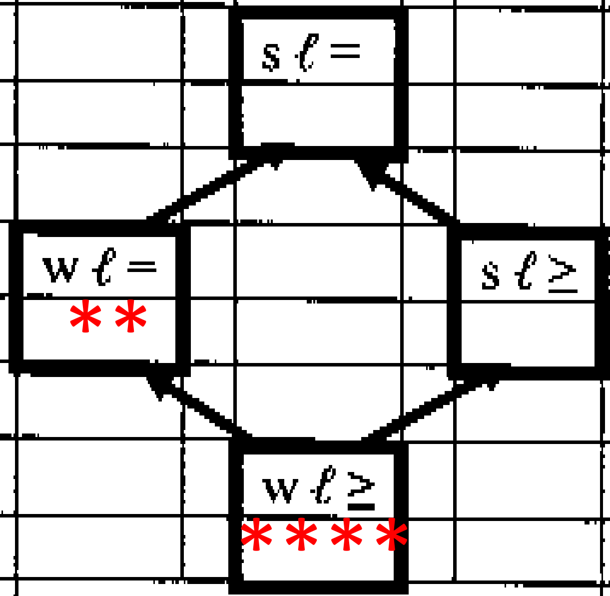

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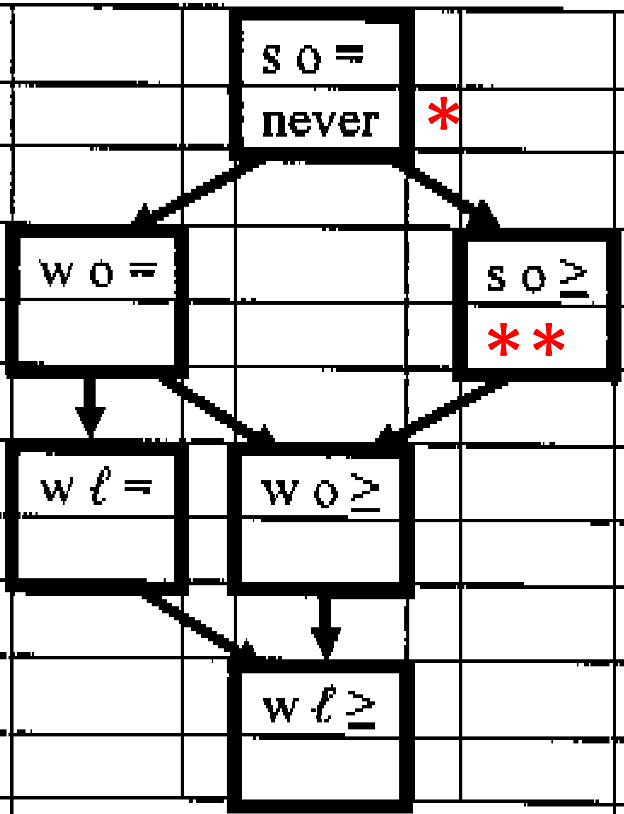

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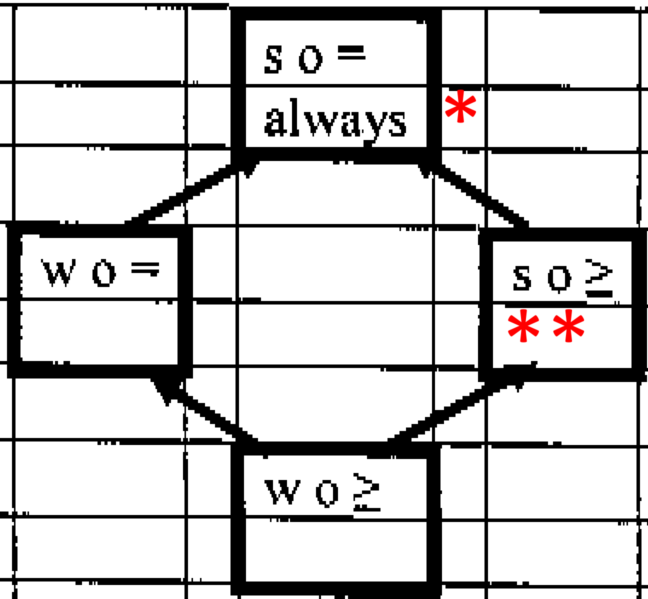

Figure 2

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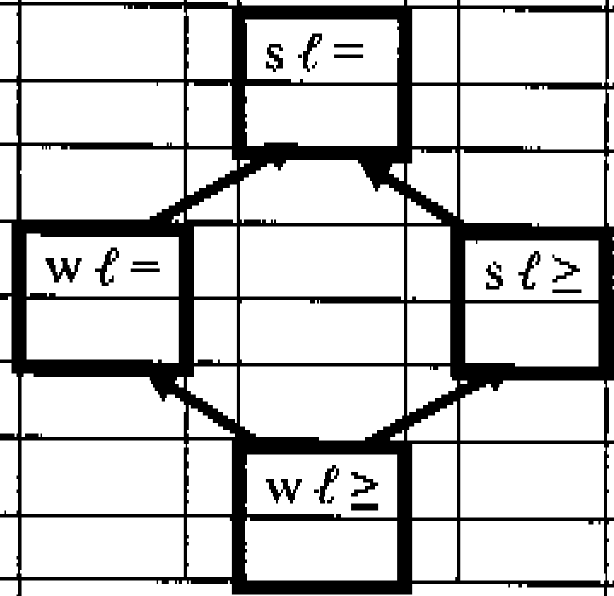

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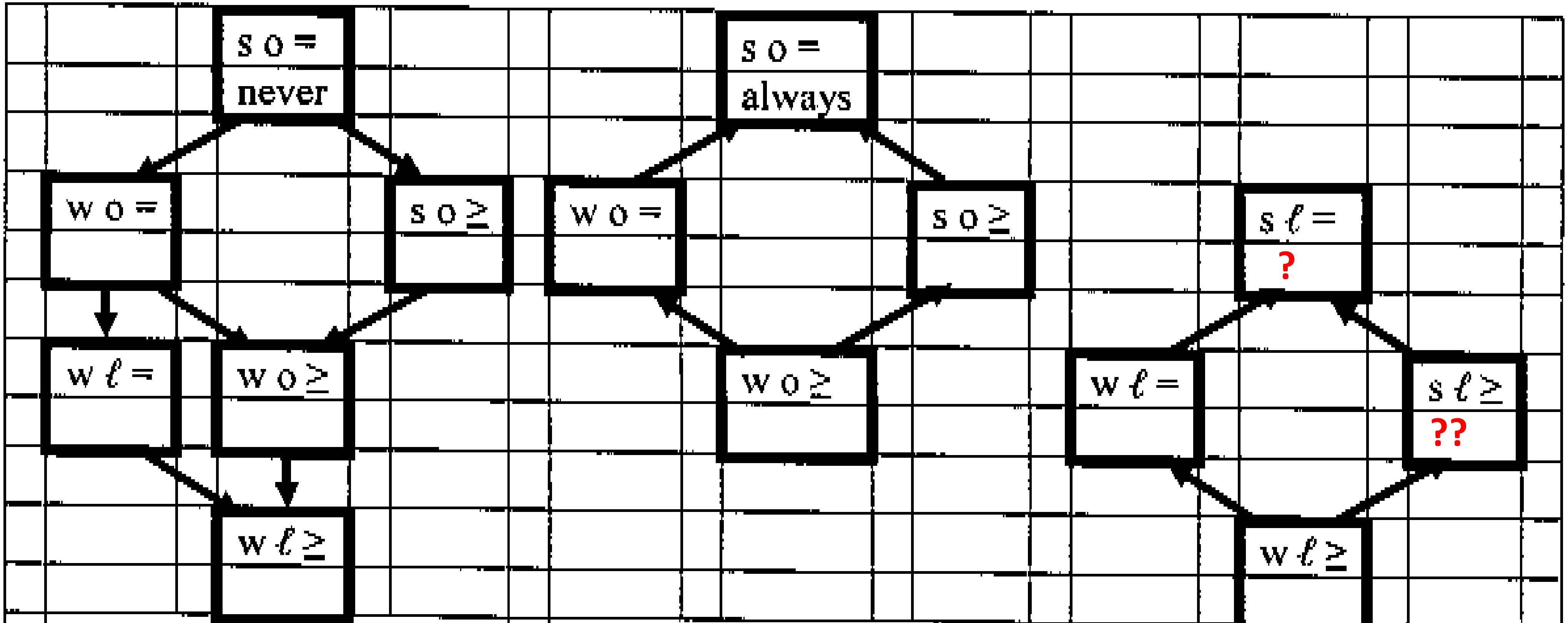


Figure 1
Subcases, solvable in polynomial time

Figure 2
Non-polynomial subcases

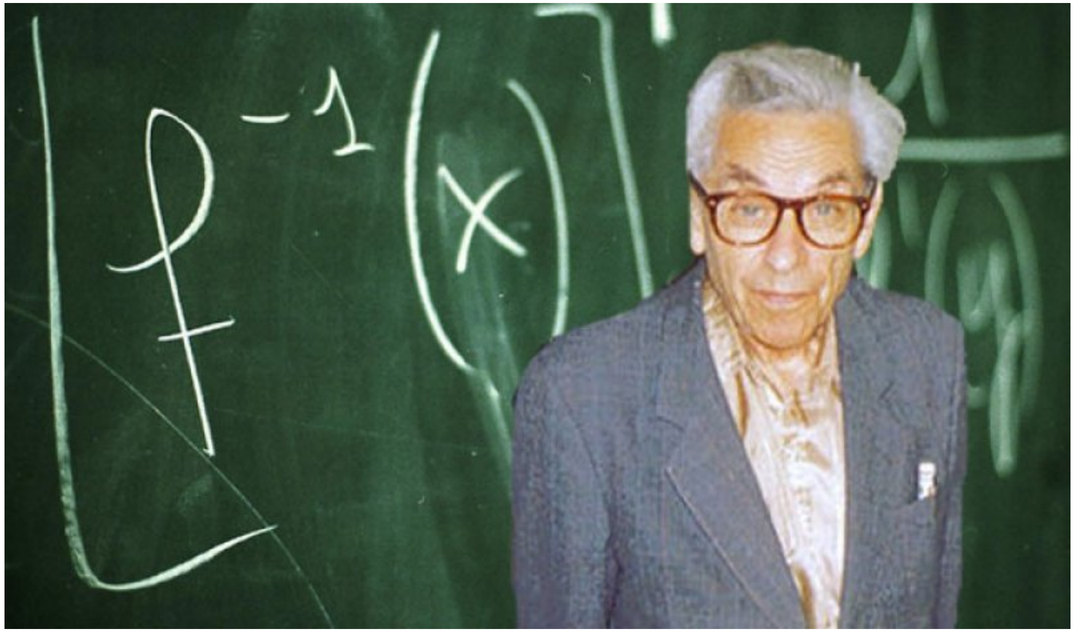
Figure 3
NP-complete subcases

Two remarks

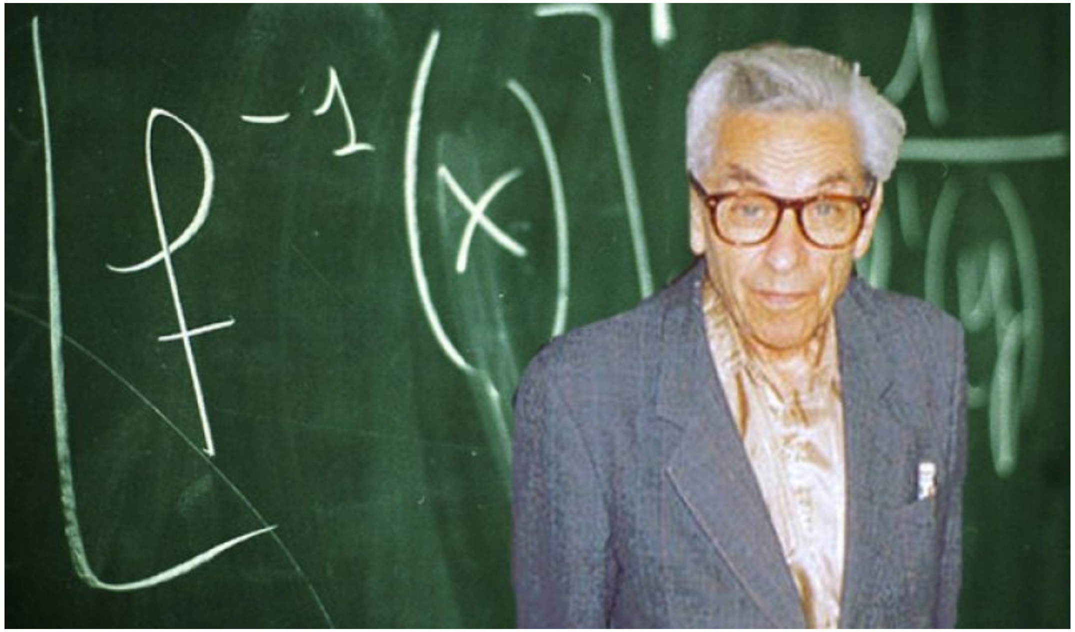
1. The simplest open cases are A = {0, 1, 3} and {0,2,3} – these cases motivated the generalization during our investigations in electric network applications 35 years ago.

2. The concept of the adjacent gaps has first appeared in the classical results of Lovász, 1970 about the generalization of Tutte’s factor theorem for graphs. However, this similarity is accidental: The case k=2, A = {0, 2} is difficult here while it is a special case of his „antifactors” which is very easy.

Back to the 1981 meeting at Luminy

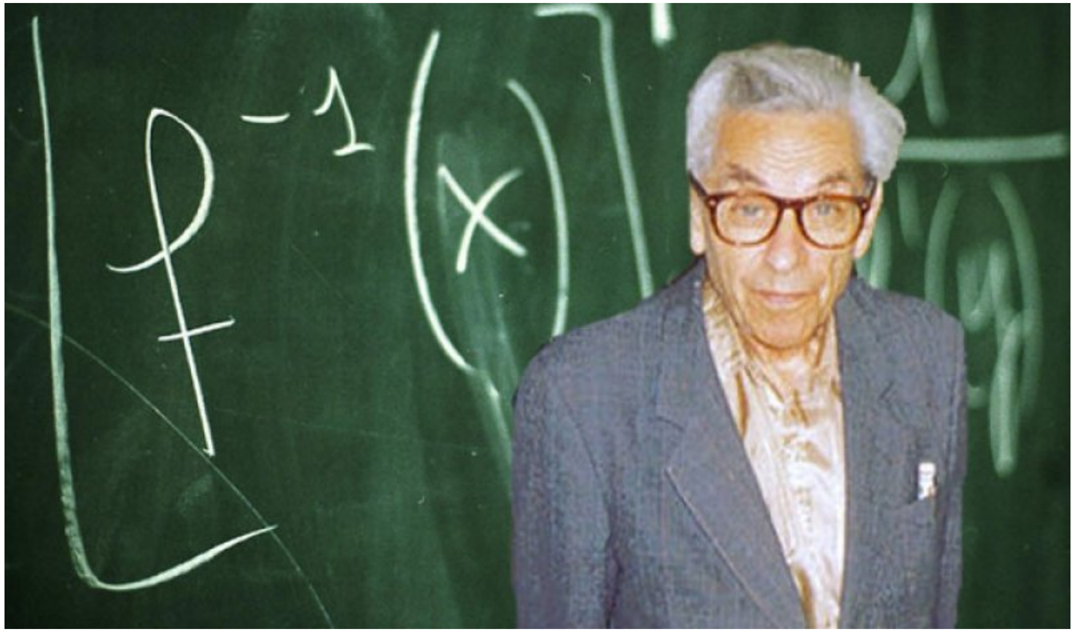


Back to the 1981 meeting at Luminy

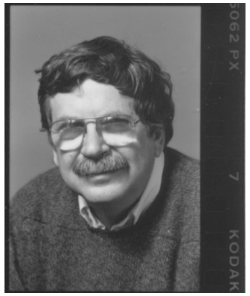
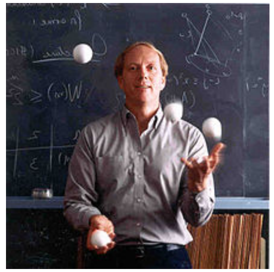
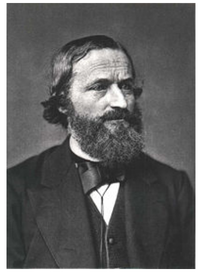


Portable immersion heater

Back to the 1981 meeting at Luminy



Portable immersion heater

Thank you very much for your attention

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