Matroids over Skew Hyperfields

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Definition (Skew hyperfield $(H, \cdot, \boxplus, 1, 0)$)

- $(H, \boxplus, 0)$ is a commutative hypergroup
- $(H^\star, \cdot, 1)$ is a ~~commutative~~ group, $H^\star := H \setminus \{0\}$
- $0 \cdot x = x \cdot 0 = 0$ for all $x \in H$
- $\alpha \cdot (x \boxplus y) = \alpha x \boxplus \alpha y$ and $(x \boxplus y) \cdot \alpha = x\alpha \boxplus y\alpha$ for all $\alpha, x, y \in H$

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Example

Skew Fields $x \boxplus y = \{x + y\}$

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Valuative Hyperfields $\Gamma_{\min} := (\Gamma \cup \{0\}, \cdot, \boxplus, 1, 0)$ where

$$x \boxplus y := \begin{cases} x & \text{if } x < y \\ y & \text{if } x > y \\ \{z \in \Gamma : z \geq x\} \cup \{0\} & \text{if } x = y \end{cases}$$

for any (bi-)ordered ~~commutative~~ group $(\Gamma, \cdot, 1, <)$

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Tropical Quaternions ... like tropical complex numbers (Viro, 2010)

Definition (Left H-matroid $M = (E, \mathcal{C})$)

E finite set, $\mathcal{C} \subseteq H^E$ such that

- (C0) $0 \notin \mathcal{C}$
- (C1) $X \in \mathcal{C}$ and $\alpha \in H^\star \implies \alpha \cdot X \in \mathcal{C}$
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- (C3) $X, Y \in \mathcal{C}$ a modular pair in $\mathcal{C}$ and $X_e = -Y_e \neq 0 \implies$ $Z_e = 0$ and $Z \in X \boxplus Y$ for some $Z \in \mathcal{C}$

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Example

Matroid over $\mathbb{K} \longleftrightarrow$ ordinary matroid

Matroid over skew field $K \longleftrightarrow$ vectors from K-linear space

Matroid over $\Gamma_{\min} \longleftrightarrow$ Γ-valuated matroid

N a matroid on E of rank r, H a commutative hyperfield.

Definition (Grassmann-Plücker function for N)

$\phi : E^r \to H$ such that

- (GP0) $\phi(B) \neq 0$ if and only if $\underline{B}$ is a basis of N.
- (GP1) $\phi(B^\tau) = \mathrm{sign}(\tau)\phi(B)$
- (GP2) $0 \in \phi(Fab)\phi(Fcd) \boxplus \phi(Fac)\phi(Fdb) \boxplus \phi(Fad)\phi(Fbc)$

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Theorem (Baker and Bowler)

An H-matroid $M = (E, \mathcal{C})$ admits a GP function ϕ for $\underline{M}$ such that

$$\phi(Fa)/\phi(Fb) = -X_b/X_a$$

whenever $X \in \mathcal{C}$ is such that $\underline{X} \subseteq \underline{F}ab$ $\qquad$ *+ vice versa.*

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GP functions for matroids over skew hyperfields $\to$

determinants for matrices over skew fields

N a matroid with bases $\mathcal{B} \to A_N := \{(B, B') \in \mathcal{B} \times \mathcal{B} : |B \setminus B'| = 1\}$.

Definition (Quasi Plücker coordinates for N)

$[.] : A_N \to H^\star$ such that

(P0) $[Fa, Fb] \cdot [Fb, Fa] = 1$

(P1) $[Fac, Fbc] \cdot [Fab, Fac] \cdot [Fbc, Fab] = -1$

(P2) $[Fa, Fb] \cdot [Fb, Fc] \cdot [Fc, Fa] = 1$

(P3) $[Fac, Fbc] = [Fad, Fbd]$ if $Fab \notin \mathcal{B}$ or $Fcd \notin \mathcal{B}$

(P4) $1 \in [Fbd, Fab] \cdot [Fac, Fcd] \boxplus [Fad, Fab] \cdot [Fbc, Fcd]$.

Theorem (P.,2018)

A left H-matroid $M = (E, \mathcal{C})$ admits a QP function $[.]$ for $\underline{M}$ such that

$$[Fa, Fb] = -X_a^{-1} X_b$$

whenever $X \in \mathcal{C}$ is such that $\underline{X} \subseteq Fab$ + *vice versa.*

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This follows from (P3) if multiplication is commutative.

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Definition (Push-forward)

If $f : H \to H'$ a hyperfield homomorphism, then the push-forward f_*M is the left H'-matroid with QP coordinates

$$[B, B']_{f_*M} := f([B, B']_M)$$

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$H^{ab} := H/\{\text{commutator subgroup of } H^\star\}$

E.g. H the skew field of quaternions $\Longrightarrow$ H^{ab} the triangle hyperfield.

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Observation

If $f : H \to H^{ab}$ the natural hom., then $f_* M$ admits a GP function.

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Observation

If $f : H \rightarrow H^{ab}$ the natural hom., then f_*M admits a GP function.

This GP function of f_*M is the Dieudonné determinant. Over H^{ab}:

$$0 \in \mathrm{Ddet}(Fab)\mathrm{Ddet}(Fcd) \boxplus \mathrm{Ddet}(Fac)\mathrm{Ddet}(Fdb) \boxplus \mathrm{Ddet}(Fad)\mathrm{Ddet}(Fbc)$$

Definition (Left circuit signature of N)

$\mathcal{C} \subseteq H^E$ and $\{\underline{X} : X \in \mathcal{C}\}$ = circuits of N, and

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$$\uparrow \qquad [Fa, Fb] = -X_a^{-1} X_b \text{ whenever } \underline{X} \subseteq Fab \qquad \downarrow$$

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Definition ($\perp_k$)

If $\mathcal{C}, \mathcal{D} \subseteq H^E$ then $\mathcal{C} \perp_k \mathcal{D} :\Longleftrightarrow$

$$0 \in \sum_e X_e \cdot Y_e$$

for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$ such that $|\underline{X} \cap \underline{Y}| \leq k$.

Theorem (P., 2018)

M a left H-matroid with circuits $\mathcal{C}$, cocircuits $\mathcal{D}$, $\underline{M} = N \Longleftrightarrow$ $\mathcal{C}$ a left H-signature of N, $\mathcal{D}$ a right H-signature of N^*, and $\mathcal{C} \perp_3 \mathcal{D}$

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Definition

M a strong left H-matroid with circuits $\mathcal{C}$, cocircuits $\mathcal{D}$, $\underline{M} = N \Longleftrightarrow$ $\mathcal{C}$ a left H-signature of N, $\mathcal{D}$ a right H-signature of N^* , and $\mathcal{C} \perp_\infty \mathcal{D}$

Mutatis mutandis: equivalent strong circuit axioms, strong QP axioms.

Definition (Double distributivity)

A skew hyperfield H is doubly distributive if

$$(a \boxplus b) \cdot (c \boxplus d) = ac \boxplus ad \boxplus bc \boxplus bd$$

for all $a, b, c, d \in H$

Conjecture

Suppose $\mathcal{C}$ left H-signature of N, $\mathcal{D}$ right H-signature of N^*.
If H is doubly distributive, then

$$\mathcal{C} \perp_3 \mathcal{D} \iff \mathcal{C} \perp_\infty \mathcal{D}$$

"Field Extensions, Derivations, and Matroids over Skew Hyperfields"

https://arxiv.org/abs/1802.02447