A matroid extension result

James Oxley

Louisiana State University

Combinatorial Geometries, Marseille-Luminy, September, 2018

Extensions

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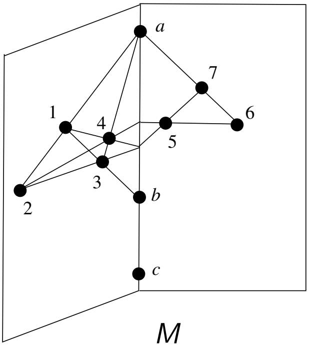

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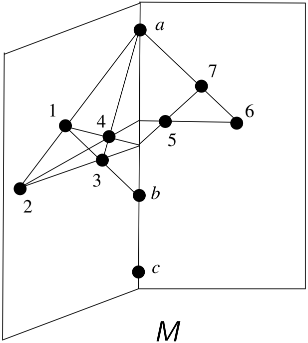
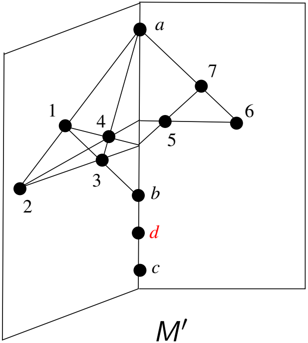


M



M'

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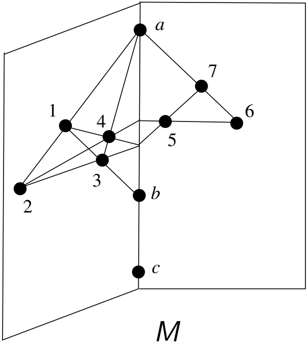

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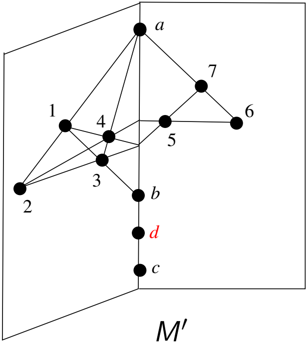

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$M' \backslash d = M$

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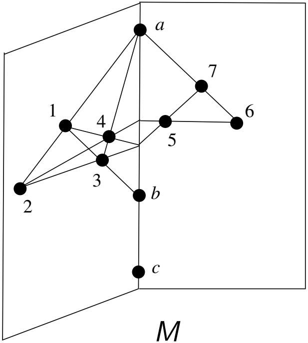

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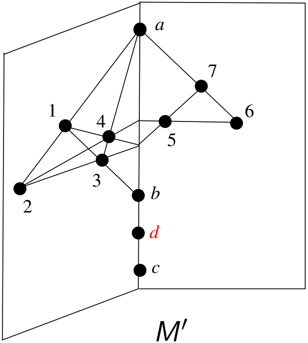

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$$M' \backslash d = M$$

A matroid M' is an extension of the matroid M by an element d if $M' \backslash d = M$.

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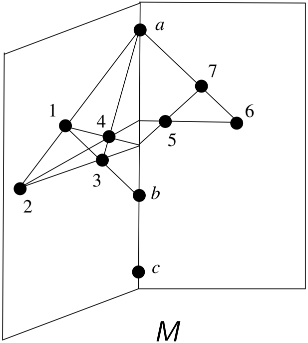

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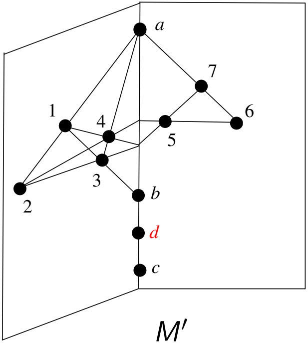

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Henry Crapo, 1965

Extending the Vámos matroid

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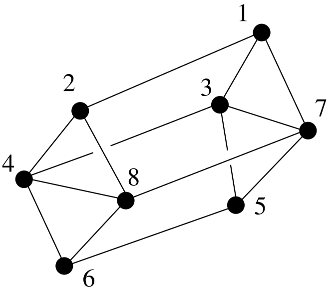

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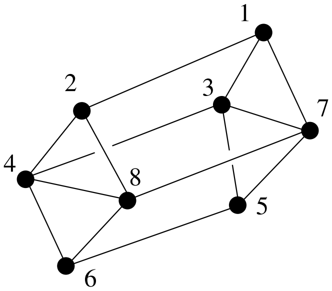

V_8 rank $= 4$

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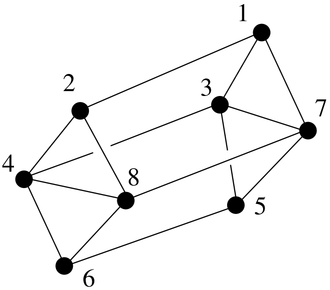

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Non-spanning circuits

$\{1,2,3,4\},\{3,4,5,6\},\{5,6,7,8\},\{7,8,1,2\},\{3,4,7,8\}$

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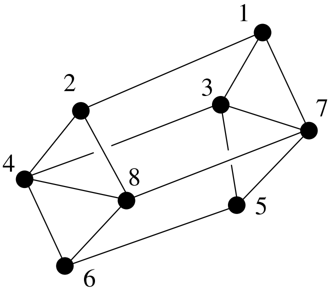

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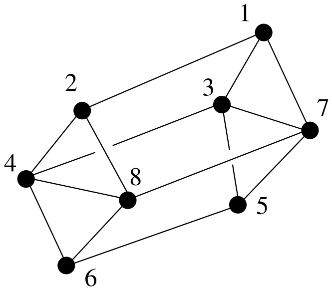

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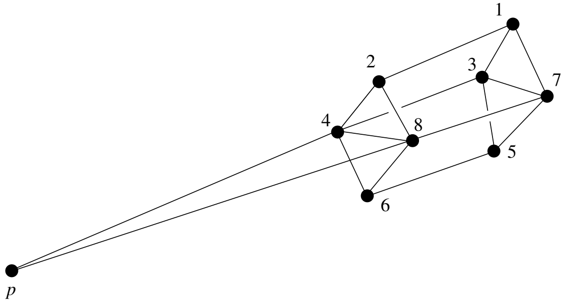

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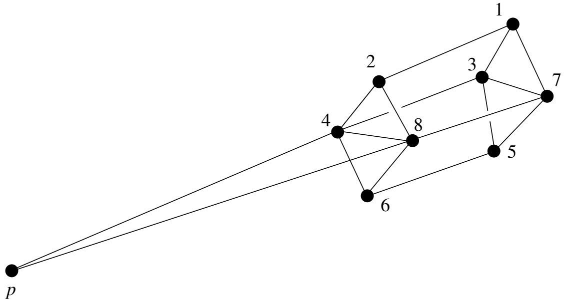

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$$

$$r(\{1,2,3,4,p\}) + r(\{5,6,7,8,p\}) \geq r(\{1,\dots,8,p\}) + r(\{1,2,p\})$$

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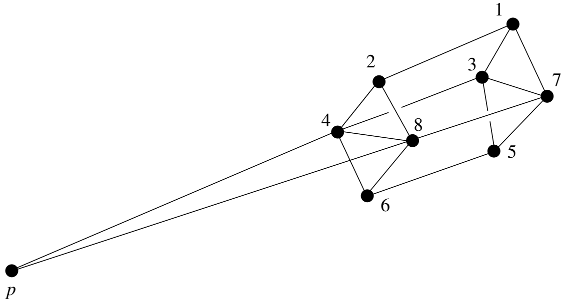

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$$r(\{1,2,p\}) = 2$$

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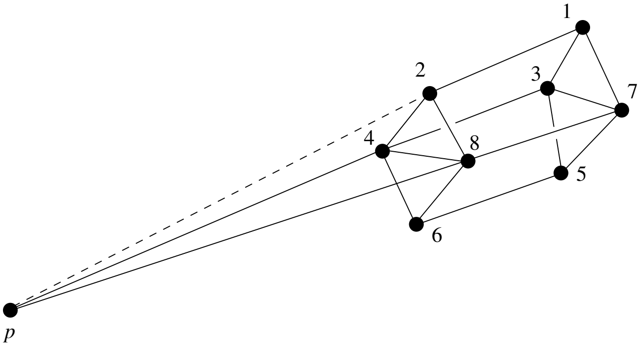

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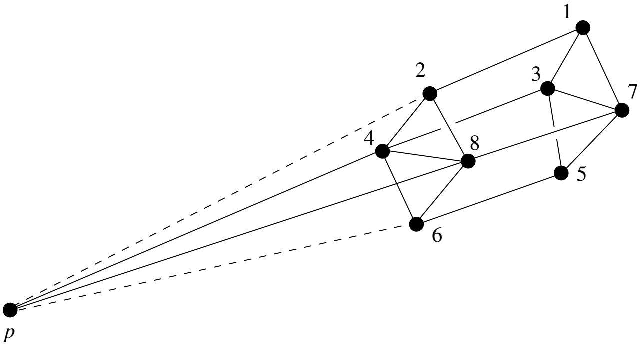

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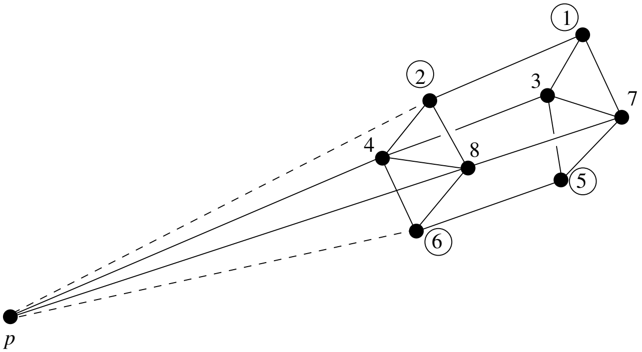

$\{1,2,5,6\}$ is NOT a basis. The extension does not exist.

3-separations

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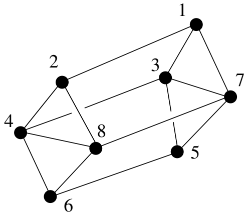

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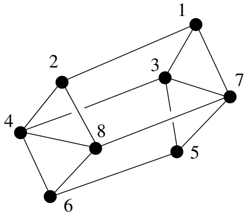

$$r(\{1,2,3,4\}) + r(\{5,6,7,8\}) - r(V_8) = 2$$

$(\{1,2,3,4\}), \{5,6,7,8\})$ is a 3-separation of $M = V_8$, that is, a partition (A, B) of $E(M)$ with

$$r(A) + r(B) - r(M) = 2 \quad \text{and} \quad |A|, |B| \geq 3$$

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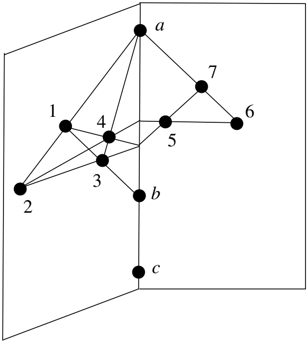

$(\{1, 2, 3, 4\}), \{5, 6, 7, a, b, c\})$ is a 3-separation of M.

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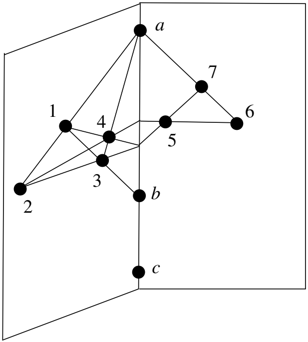

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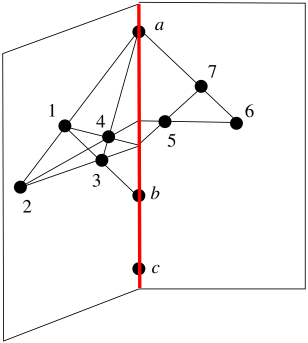

- $\{a, b, c\}$ is the guts line of the 3-separation.

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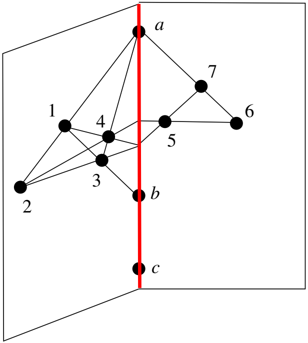

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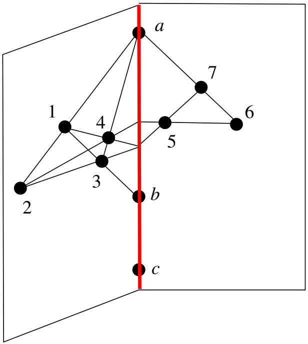
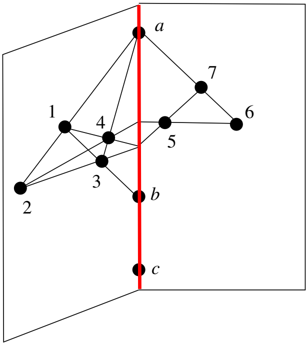


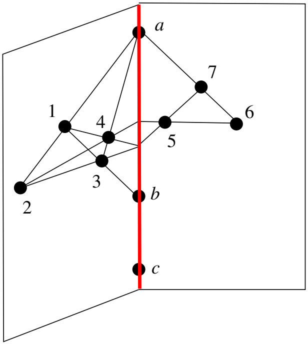
- $\{a, b, c\}$ is the guts line of the 3-separation.
- a and b are fixed.
- c is not fixed.

3-separations



- $\{a, b, c\}$ is the guts line of the 3-separation.
- a and b are fixed.
- c is not fixed. It is freely placed on the guts line.

Fixed elements



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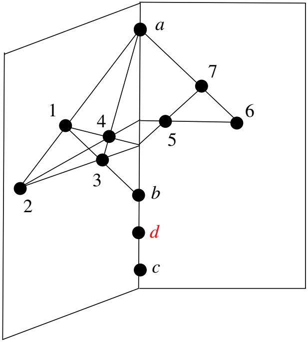

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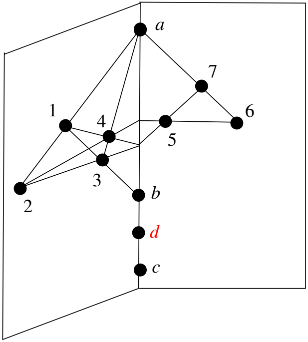

- c and d are independent clones.

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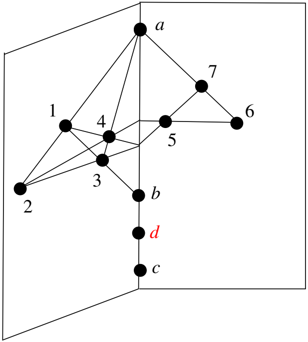

- c and d are independent clones.

The only way to add a clone of b is to add it parallel to b.

Freely adding elements to the guts line of a 3-separation

Theorem (Geelen, Gerards, Whittle; 2006)

Given a 3-separation (A, B) in a matroid M, there is a unique extension N of M by independent clones x and y so that both are freely placed on the guts line of (A, B).

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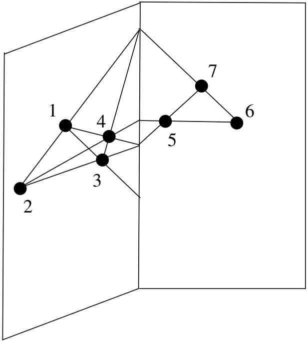

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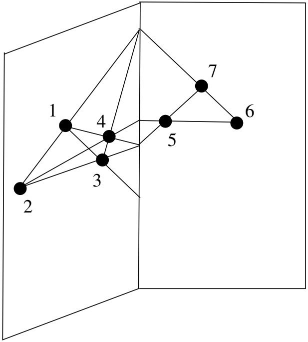

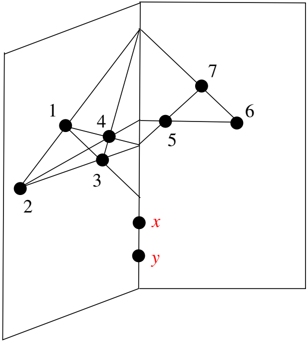

Local connectivity

For sets X and Y in M, the local connectivity is

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

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When M is a subset of a vector space,

$$\sqcap(X, Y) = \dim(\langle X \rangle \cap \langle Y \rangle).$$

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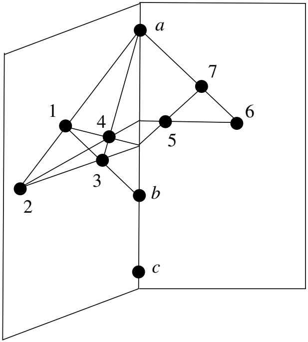

$(A, B) = (\{1, 2, 3, 4\}, \{5, 6, 7, a, b, c\})$ is a 3-separation.

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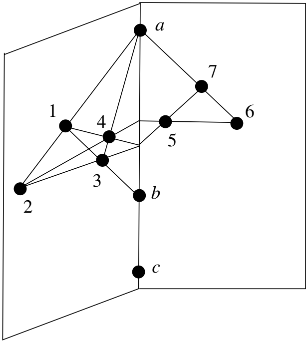

$$(A, B) = (\{1, 2, 3, 4\}, \{5, 6, 7, a, b, c\}) \quad \text{is a 3-separation.}$$

- $\{1, 2\}, \{1, 4\}$ and $\{2, 3\}$ are *A*-strands,

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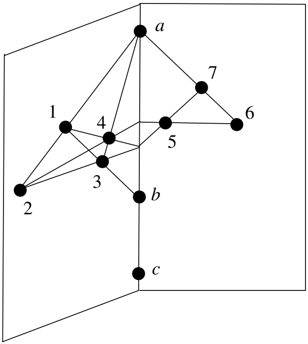

- $\{1, 2\}, \{1, 4\}$ and $\{2, 3\}$ are *A-strands*, that is, minimal subsets X of A such that $\sqcap(X, B) = 1$.

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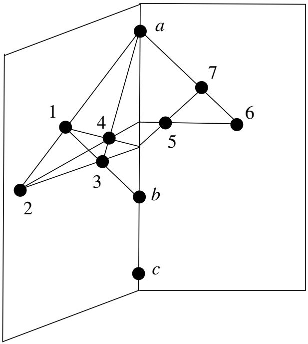
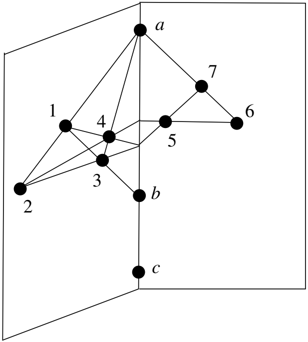


- $\{1,2\}, \{1,4\}$ and $\{2,3\}$ are *A*-strands, that is, minimal subsets X of A such that $\sqcap(X, B) = 1$.
- $\{5,7\}$ and $\{a\}$ are examples of B-strands.

Strands



- $\sqcap(\{1,4\},\{5,7\}) = 1.$

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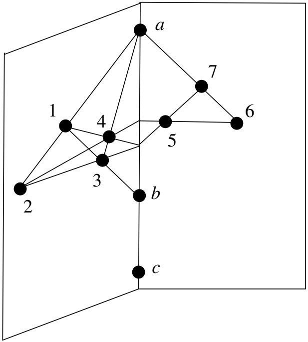

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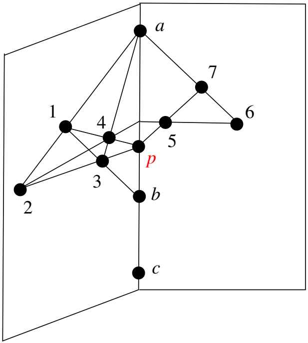

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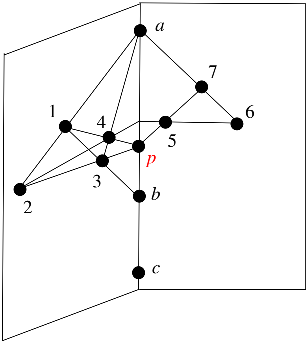
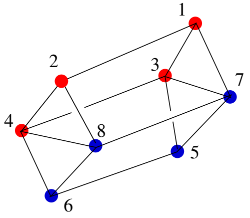


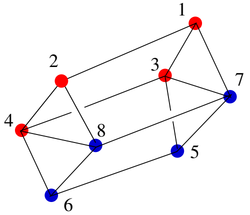
This forces $\{2,3,p\}$ to be a circuit.

Back to the Vámos



$$(A, B) = (\{1, 2, 3, 4\}, \{5, 6, 7, 8\}) \quad \text{is a 3-separation.}$$

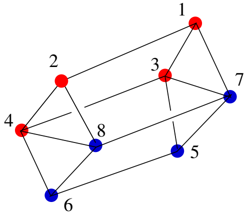
Back to the Vámos



$$(\color{red}{A}, \color{blue}{B}) = (\color{red}{\{1,2,3,4\}}, \color{blue}{\{5,6,7,8\}}) \quad \text{is a 3-separation.}$$

- $\color{red}{\{1,2\}}$ and $\color{red}{\{3,4\}}$ are $\color{red}{A}$-strands.

Back to the Vámos



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- $\{5, 6\}$ and $\{7, 8\}$ are B-strands.

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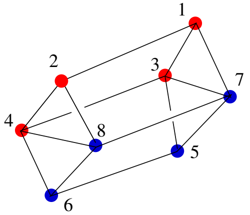

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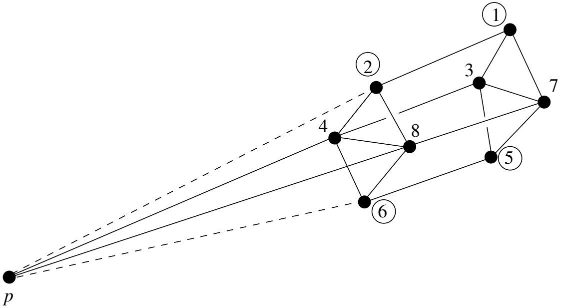
- $\{5, 6\}$ and $\{7, 8\}$ are B-strands.
- $\sqcap(\{3, 4\}, \{7, 8\}) = 1$.

Back to the Vámos



$(A, B) = (\{1, 2, 3, 4\}, \{5, 6, 7, 8\})$ is a 3-separation.

- $\{1, 2\}$ and $\{3, 4\}$ are A-strands.
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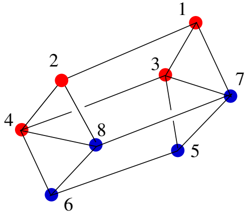
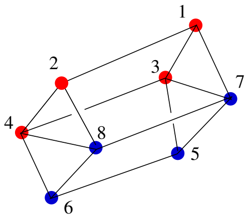
We CANNOT add p so that $\{3, 4, p\}$ and $\{7, 8, p\}$ are circuits.

Back to the Vámos



$$({\color{red}A}, {\color{blue}B}) = ({\color{red}\{1, 2, 3, 4\}}, {\color{blue}\{5, 6, 7, 8\}}) \quad \text{is a 3-separation.}$$

Back to the Vámos



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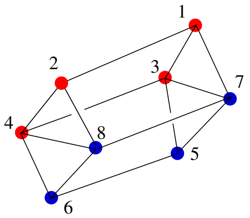

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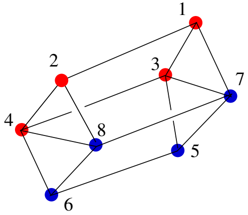
Back to the Vámos



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- $\sqcap(\{1,2\},\{5,6\}) = 0$

Main theorem

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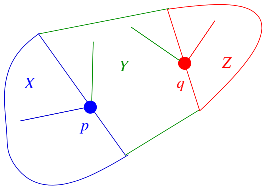
The Vámos matroid encapsulates the core obstruction to this extension.

Multiple extensions



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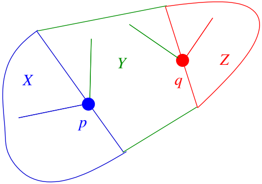
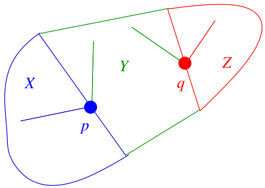


Theorem

Let (X, Y, Z) be a partition of a matroid where Y may be empty. Let $(X, Y \cup Z)$ and $(X \cup Y, Z)$ be 3-separations.

- *Assume X_0 and Y_0 are an X-strand and a $(Y \cup Z)$-strand with $\sqcap(X_0, Y_0) = 1$ such that M has an extension by p so that $X_0 \cup p$ and $Y_0 \cup p$ are circuits.*

Multiple extensions



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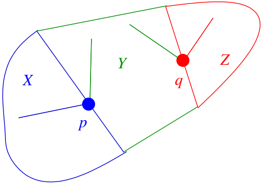
- *Assume Y_1 and Z_1 are an $(X \cup Y)$-strand and a Z-strand with $\sqcap(Y_1, Z_1) = 1$ such that M has an extension by q so that $Y_1 \cup q$ and $Z_1 \cup q$ are circuits.*

Multiple extensions



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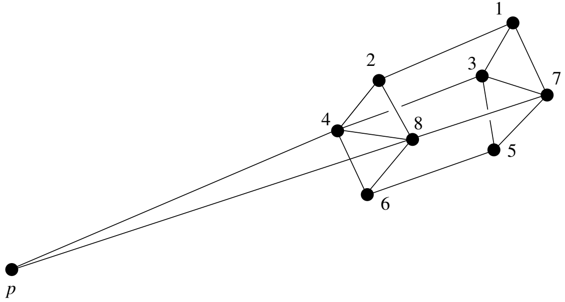
Then M has a unique extension by p and q such that all of $X_0 \cup p$, $Y_0 \cup p$, $Y_1 \cup q$, and $Z_1 \cup q$ are circuits.

The Vámos matroid is the key obstruction



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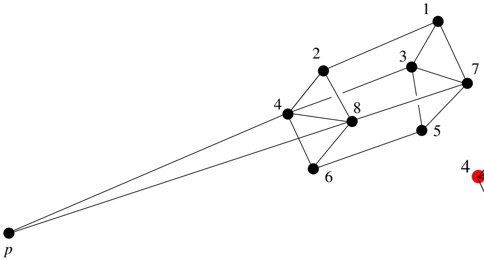

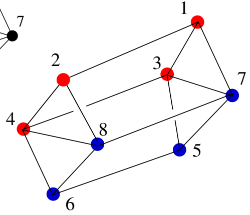

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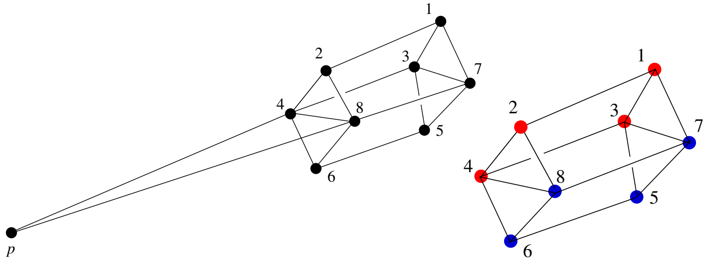

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