Excluded Minor Results For Vf-Safe Delta–Matroids

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Quick recap 1.

Definition (Bouchet)

$(E, \mathcal{F})$ form a Δ-matroid if

1. $\mathcal{F} \neq \emptyset$.
2. For all $F_1, F_2 \in \mathcal{F}$ and $e \in F_1 \triangle F_2$, there exists $f \in F_1 \triangle F_2$ such that $F_1 \triangle \{e, f\} \in \mathcal{F}$.

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$\mathcal{F}$ is the set of feasible sets.

The set of quasi-trees of a ribbon graph form the feasible sets of a ribbon-graphic Δ-matroid.

Quick recap 2.

Let $D = (E, \mathcal{F})$ be a Δ-matroid and $A \subseteq E$. Then the partial dual $D * A$ is the Δ-matroid such that F is feasible in $D * A$ if and only if $F \triangle A$ is feasible in D.

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Partial (local) duality of ribbon graphs is consistent with that of Δ-matroids.

Binary Δ-matroids 1.

A symmetric binary matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The non-singular principal sub-matrices

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The collection of (sets of column indices of) non-singular, principal sub-matrices of a symmetric binary matrix M form the feasible sets of a Δ-matroid $D(M)$.

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These Δ-matroids are (confusingly) called graphic.

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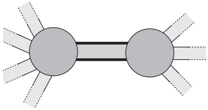
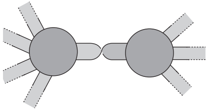
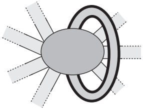
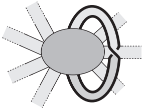
1. *Every normal binary Δ-matroid is graphic.*
2. *Every binary matroid is a binary Δ-matroid.*
3. *Every ribbon-graphic Δ-matroid is binary.*

Half-Twists

A natural ribbon graph operation is the partial Petrial: apply a half-twist to an edge e to get $G^{\tau(e)}$.

G	$G^{\tau(e)}$
	
	

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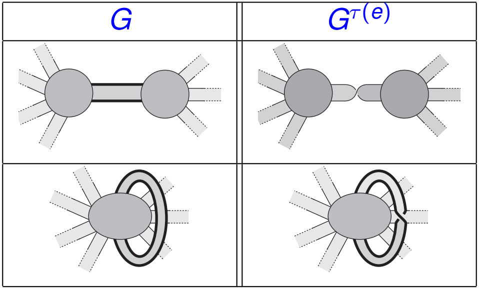

Extends to sets of edges: $G^{\tau(A)}$ denotes G with half-twists added to the edges of A.

Loop complementation

If M is a symmetric binary matrix, then $M + e$ is formed from M by replacing $M_{e,e}$ by $1 - M_{e,e}$.

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Proposition (Brijder, Hoogeboom)

1. *If $e \notin F$, then $F \in \mathcal{F}(D(M + e)) \Leftrightarrow F \in \mathcal{F}(D(M))$.*
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Proposition (Chun, Moffatt, N, Rueckriemen)

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Open Question

What are the excluded minors for the class of vf-safe Δ-matroids?

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Theorem

A matroid is vf-safe if and only if it has no minor isomorphic to $U_{2,6}$, $U_{4,6}$, P_6, F_7^- or $(F_7^-)^*$.

Vf-safe Δ-matroids

The operations $+e$ and $*e$ act on set systems like two transpositions in Sym_3.

Definition (Brijder and Hoogeboom)

If a set system S' may be obtained from S by a sequence of $+$ and $*$ operations, then S' is a twisted dual of S.

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- All three types of minor can be defined for arbitrary set systems.
- The order of operations matters, but not if we take care to avoid contracting loops, deleting coloops etc.
- So there is a well-defined notion of 3-minors.

Vf-safe Δ-matroids

Theorem (Bonin, C. Chun, N)

A set system is a vf-safe Δ-matroid if and only if it contains no 3-minor isomorphic to a twisted dual of S_3.

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Proof.

Each member of $\mathcal{S} \cup \mathcal{T}$ contains a twisted dual of S_3 as a 3-minor. $\square$

Vertex minors 1.

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Two graphs are locally equivalent if one can be obtained from the other by a sequence of the following:

- adding / removing loops (loop complementations).
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G is a vertex minor of H if G may be obtained from H by a sequence of local / loop complementations and deleting vertices.

Vertex minors 2.

"Vertex minors and 3-minors are equivalent".

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Proposition

- *If G is a vertex minor of H, then $D(G)$ is a twisted dual of a 3-minor of $D(H)$.*
- *If D_1 is a 3-minor of a binary Δ-matroid D_2, then there are graphs G_1 and G_2 with $D(G_i)$ a twisted dual of D_i and G_1 a vertex minor of G_2.*

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Up to partial duals there are 5 excluded minors for the class of binary Δ-matroids within the class of Δ-matroids.

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Corollary (Bonin, Chun, N)

A vf-safe Δ-matroid is binary if and only if it has no 3-minor isomorphic to a twisted dual of

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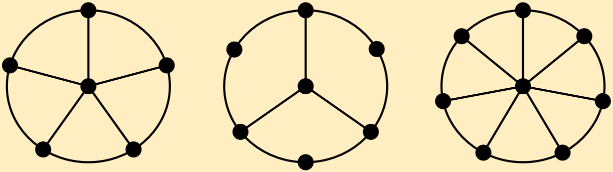
A set system is a binary Δ-matroid if and only if it has no 3-minor isomorphic to a twisted dual of S_3 or B.

Ribbon graphs

Theorem (Bouchet)

G is a circle graph (i.e. a ribbon graph with one vertex) if and only if G has no vertex minor isomorphic to one of



G_1 G_2 G_3 .

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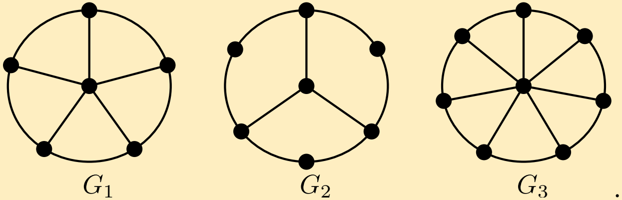

Theorem (Bonin, Chun, N)

A set system is a ribbon graphic Δ-matroid if and only if it has no 3-minor isomorphic to a twisted dual of S_3, B_1, $D(G_1)$, $D(G_2)$ or $D(G_3)$.