GRADED BIALGEBRAS AND DELETION-CONTRACTION INVARIANTS

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joint work with

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Combinatorial Geometries

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Summary

1. Multiplicative minors systems
2. Grothendieck monoid and universal Tutte character
3. Applications to graphs, matroids, arithmetic matroids
4. Other applications (sketch).

1. MULTIPLICATIVE MINORS SYSTEMS

Let $\mathbb{K}$ be a commutative ring with 1.

We recall that an (associative) **algebra** is a $\mathbb{K}$-module A with $\mathbb{K}$-linear maps $\mu : A \otimes A \to A$ ("product") and $\eta : \mathbb{K} \to A$ ("unit") satisfying the usual conditions.

A (coassociative) **coalgebra** is a $\mathbb{K}$-module A with $\mathbb{K}$-linear maps $\Delta : A \to A \otimes A$ ("coproduct") and $\varepsilon : A \to \mathbb{K}$ ("counit") satisfying dual conditions.

A **bialgebra** is a $\mathbb{K}$-module A with $\mathbb{K}$-linear maps $\mu, \eta, \Delta, \varepsilon$ that make A into both an algebra and a coalgebra and that are compatible.

A bialgebra A is called a **Hopf algebra** if there exists $a \in \mathrm{End}(A)$ (the "antipode") such that the diagram

$$\begin{tikzcd}
& A\otimes A \arrow[r, "a\otimes id"] & A\otimes A \arrow[dr, "\mu"] & \\
A \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \mathbb{K} \arrow[rr, "\eta"] & & A \\
& A\otimes A \arrow[r, "id\otimes a"'] & A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

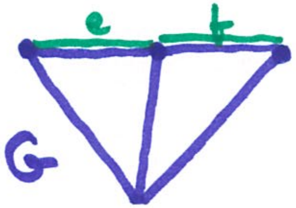
commutes.

<u>Example</u> Let $(G, \cdot)$ be a group. The $\mathbb{K}$-module free on elements $\{t_g, g \in G\}$ is an algebra with product extending linearly $\mu(t_g \otimes t_h) = t_{g \cdot h}$ and unit $\eta(1) = t_e$.
It is a coalgebra (and a bialgebra) for $\Delta(t_g) = t_g \otimes t_g$, $\varepsilon(t_g) = 1$ $\forall g \in G$.
Moreover it is a Hopf algebra with antipode $a(t_g) = t_{g^{-1}}$ $\forall g \in G$

Example to keep in mind in the next 10 minutes:

- S is the set of all isomorphism classes of graphs
- $\forall G \in S$, $E(G)$ is the set of edges of G
- $\forall A \subseteq E(G)$, $G|_A$ is the restriction and G/A is the contraction

Example:

G 

$A = \{e, f\}$

$G|_A$ 

G/A 

- $\forall G_1, G_2 \in S$, $G_1 \oplus G_2$ is the disjoint union.

(Lexical problem: the contraction is called "restriction" by people in hyperplane arrangements...)

Remark: the restriction $G|_A$ is the deletion of the complement $A^c := E(G) \setminus A$.

Def A multiplicative minors system (MMS) is a set S with:

- $\forall G \in S$, a finite set $E(G)$
- $\forall A \subseteq E(G)$, $G|_A, G/A \in S \mid E(G|_A) = A$, $E(G/A) = A^c$
- $\forall G_1, G_2 \in S$, $G_1 \oplus G_2 \in S \mid E(G_1 \oplus G_2) = E(G_1) \sqcup E(G_2)$

satisfying the following compatibilities:

1) $G|_{E(G)} = G = G/\emptyset$
2) $\forall A, B \subseteq E(G) \mid A \cap B = \emptyset$, $G/(A \cup B) = (G/A)/B$, $G|_{(A\cup B)^c} = (G|_{A^c})|_{B^c}$, $(G|_{A^c})/B = (G/B)|_{A^c}$
3) $\oplus$ is ~~asso~~ associative and commutative, with a neutral element 0_S.
4) $\forall G_1, G_2 \in S$, $\forall A_1 \subseteq E_1$, A_2 ~~$\subseteq E_2$, $(G_1 \oplus G_2)$~~ $/(A_1 \sqcup A_2) = (G_1/A_1) \oplus (G_2/A_2)$ and the same for restriction.

Def Morphism of MMS: a function $f: S \to S'$ compatible with E, $|$, $/$, $\oplus$.

Let S be a MMS and $\mathbb{K}S$ be the free module on S. It is naturally graded by $\mathbb{K}S = \bigoplus_{n \in \mathbb{N}} \mathbb{K}S_n$, where $S_n = \{G \in S \,|\, |E(G)| = n\}$.

$\mathbb{K}S$ has product $\mu(G_1 \otimes G_2) = G_1 \oplus G_2$, with unit 0_S, and coproduct $\Delta(G) = \sum_{A \subseteq E(G)} G|_A \otimes G/A$ with counit $\varepsilon(G) = \begin{cases} 1 & \text{if } G \in S_0 \\ 0 & \text{otherwise.} \end{cases}$

Prop $\mathbb{K}S$ is a graded ~~$\mathbb{K}$-bi~~algebra, and $S \mapsto \mathbb{K}S$ is a functor.

The proof is straightforward: the operations respect the grading, and are compatible by axioms 1 and 4. Axioms 2 and 3 yield coassociativity and associativity resp. Functoriality is also easy to check.

Def: A graded bialgebra A is connected if $A_0 \simeq \mathbb{K}$

Fact: In this case A is Hopf, with the antipode given by an explicit formula (by Takeuchi).

Example: $S = \{\text{graphs}\}$. $\mathbb{K}S$ is not connected because
$S_0 = \{\text{graphs with no edges}\} = \{\cdot, \cdot\cdot, \cdot\cdot\cdot, \dots\} \simeq \mathbb{N}$
$\Rightarrow \mathbb{K}S_0$ is the algebra $\mathbb{K}[x] \subseteq \mathbb{K}[x, x^{-1}] \simeq \mathbb{K}[\mathbb{Z}]$ with $\Delta(x) = x \otimes x$,
so it has no antipode (it should be $x \mapsto x^{-1}$).
So $\mathbb{K}S_0$ is not Hopf $\Rightarrow$ $\mathbb{K}S$ is not Hopf!

Rem: We may obtain a Hopf algebra as a quotient of $\mathbb{K}S$.
However, we will see that it is more convenient not to lose the information contained in S_0, and to work with the whole bialgebra $\mathbb{K}S$.

A bit of history:

- '79 Joni – Rota: several combinatorial objects have a bialgebra structure
- '94 Schmitt: many of them are Hopf (structure described, antipode computed)
- '15 Krajewski – Moffatt – Tanasa: several invariants can be computed from the Hopf algebra
- '17 Dupont – Fink – M.: more invariants can be computed from the bialgebra $\mathbb{K}S$

In fact we need even less: the coalgebra plus a small piece of the product, corresponding to $G \oplus U$ with $U \in S_0$.
This is useful for dealing with those "minors system" for which $\oplus$ is not defined in general!

2. GROTHENDIECK MONOID AND UNIVERSAL TUTTE CHARACTER.

Def Let S be a MMS and X be a commutative monoid.

A **norm** is a function $N: S \to X$ satisfying:

1) $N(G) = N(G|_A)\, N(G/A)$ $\forall G \in S,\ A \subseteq E(G)$
2) $N(U \oplus G) = N(G)$ $\forall U \in S_0,\ G \in S$.
3) $N(0_S) = 1_X$.

(Those imply $N(G_1 \oplus G_2) = N(G_1) \cdot N(G_2)$).

Def The **Grothendieck monoid $X(S)$** is the commutative monoid with generators $\{[G],\ G \in S\}$ and relations:

$[G] = [G|_A][G/A]$, $[U \oplus G] = [G]$, $[0_S] = 1$.

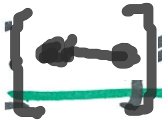
Rem: Clearly a morphism of monoids $X(S) \to S$ is equivalent to a norm for S with values in X.

Then the map
$$\begin{aligned} S &\to X(S) \\ G &\mapsto [G] \end{aligned}$$
is called the **universal norm**

Theor (DFM) $X(S)$ is generated by the classes $[G]$, $G \in S_1$, with relations:

1) $\forall G \in S_1$, $V \in S_0$, $[V \oplus G] = [G]$

2) $\forall G \in S_2$, let $E(G) = \{e, f\}$; then $[G|_e][G/_e] = [G|_f][G/_f]$

Example: for $S = \{\text{graphs}\}$, $X(S)$ is generated by the two elements  $= u$ and  $= v$. Relations are trivial, so $\mathbb{k}[X(S)] \simeq \mathbb{k}[u, v]$

In order to define the Tutte character, we still need two definitions. Recall that if (C, Δ, ε) is a coalgebra and (A, μ, η) an algebra, the convolution of two linear maps $f, g : C \to A$ is

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

$$A \leftarrow A \otimes A \leftarrow C \otimes C \leftarrow C$$

This makes $\mathrm{Hom}_{\mathbb{k}}(C, A)$ into an associative algebra with unit $u = \eta \circ \varepsilon$.

Def Let S be a MMS and X be a monoid.

A **twist map** is a morphism of monoids $\tau : S_0 \to X$.

It extends linearly to $\mathbb{K}S_0$, then to $\mathbb{K}S$ by setting $\tau(G) = 0$ $\forall G$ S_n, $n > 0$.

Let R be a commutative algebra (which we identify with is multiplicative monoid), $N_1, N_2 : \mathbb{K}S \to R$ two norms, $\tau : \mathbb{K}S \to R$ a twist map.

Def The **Tutte character** $T_{(N_1, \tau, N_2)}$ is the convolution product

$T_{(N_1, \tau, N_2)} := N_1 * \tau * N_2 : \mathbb{K}S \to R$.

"character" because is a morphism of algebras ($*$ of morphisms)

why "Tutte"? We have a deletion-contraction formula:

Prop $\forall G \in S$, $\forall e \in E(G)$, $T_{(\ )}(G) = N_1(G|_e)\, T_{(\ )}(G/e) + N_2(G/e^c)\, T_{(\ )}(G|_{e^c})$

(Sketch of proof: write $T_{(\ )}(G)$ as $\sum_{A \subseteq E(G)} N_1(G|_A)\, \tau(G|_A/A)\, N_2(G/A)$,

then break this sum into $A \ni e$, $A \not\ni e$.)

Def The universal Tutte character T_S of a MMS S is the Tutte character associated with two copies of the universal norm and $\tau : \mathbb{K}S \to \mathbb{K}S_0$ the natural projection.

Why "universal"?

Prop Let R be a commutative $\mathbb{K}$ algebra and $\Phi : \mathbb{K}S \to R$ be a linear map such that $\Phi(G) = N_1(G/e)\,\Phi(G/e) + N_2(G/e^c)\,\phi(G/e^c)$ for some norms N_1, N_2. Then there exists $\overline{\Phi}$ such that

$$\begin{tikzcd}
\mathbb{K}[S] \arrow[d, "T_S"] \arrow[dr, "\Phi"] & \\
\mathbb{K}[X(S)\times S_0 \times X(S)] \arrow[r, "\overline{\Phi}"] & R
\end{tikzcd}$$

commutes.

An easy computation shows that if $N : \mathbb{K}S \to R$ is a norm, then $\overline{N}(G) := (-1)^{|E(G)|} N(G)$ is its inverse for the convolution: $N * \overline{N} = \overline{N} * N = u$.

<u>Theor</u> (DFM) Let R be a commutative algebra, N_1, N_2 and N be three norms, and T_1, T_2 be to twist maps. Then

$$T_{(N_1, T_1 T_2, N_2)} = T_{(N_1, T_1, N)} * T_{(\overline{N}, T_2, N_2)}$$

"convolution formula".

In particular when N_1, N_2 and N are the universal norm and T_1, T_2 are the natural projection $\mathbb{K}S \to \mathbb{K}S_0$, we get a "universal convolution formula".

We will see that many convolution formulae for combinatorial invariants, old and new, can be obtained by specializing this one.

3. APPLICATIONS TO GRAPHS, MATROIDS AND ARITHMETIC MATROIDS

Example 0: $S = \{\text{sets}\}$ Trivial example, final object in category MHS

Example 1: $S = \{\text{graphs}\}$ We saw that $S_0 \simeq \mathbb{N} \simeq a^{\mathbb{N}}$

(where $k(G)$ = number of connected components) $G \longmapsto a^{k(G)}$

and that $X(S)$ is generated by $u = [$  $]$, $v = [$  $]$.

More precisely by the presentation theorem:

$$X(S) \xrightarrow{\sim} u^{\mathbb{N}} v^{\mathbb{N}}$$

$$[G] \longmapsto u^{rk(G)} v^{cork(G)}$$

, where $rk(G) = |V(G)| - k(G)$ and $cork(G) = |E(G)| - rk(G)$.

The universal Tutte character is

$$T_S : N_1 * \mathcal{T} * N_2 : \mathbb{K} S \longrightarrow \mathbb{K}[u_1, v_1, a, u_2, v_2]$$

$$G \longmapsto \sum_{A \subseteq E(G)} u_1^{rk(G/A)} v_1^{cork(G/A)} a^{k(G/A/A)} u_2^{rk(G/A)} v_2^{cork(G/A)}$$

This 5-variables polynomial specializes to some well-known 2 variable polynomials:

$$T_S(1, y-1, 1, x-1, 1) = T_G(x,y) := \sum_{A \subseteq E(G)} (x-1)^{\mathrm{rk}\,E - \mathrm{rk}\,A} \, (y-1)^{|A| - \mathrm{rk}\,A}$$ "Tutte polynomial"

$$T_S(1, y, x, 1, 1) = D_G(x,y) := \sum_{A \subseteq E(G)} x^{|V(G)|} \, y^{\mathrm{cork}\, G}$$ "Dichromate polynomial"

Historical digression: Tutte introduced D_G as a polynomial specializing to both the chromatic polynomial $\chi_G(q) = (-1)^{\mathrm{rk}(G)} D_G(-q, 0)$ and the flow polynomial $\varphi(q) = (-1)^{\mathrm{rk}(G)} D_G(0, -q)$, and such that $D_{G^*}(x, y) = D_G(y, x)$ for every planar G. Then T_G arose as a variation of D_G. T_G is an invariant of the matroid associated with G, while D_G isn't...

By the specialization above, the universal convolution formula for Ts specializes to the following

Theor (Etienne-Las Vergnas '98, Kook-Reiner-Stanton '99)

$$T_G(x,y) = \sum_{A \subseteq E(G)} T_{G/A}(0,y)\, T_{G/A}(x,0)$$

It also specializes to new formulae, for instance:

$$D(x_1, x_2, y) = \sum_{A \subseteq E} D_{G/A}(x_1, y)\, \chi_{G/A}(x_2).$$

or many others... Just plug your favourite values into the five variables!

Example 2: $S = \{\text{matroids}\}$ $\quad S \ni M = (E, rk)$, $rk : 2^E \to \mathbb{N}$ (axioms).

Similar but now $|S_0| = 1$, so $\mathbb{k}S$ is connected $\Rightarrow$ Hopf.

Morphism $S_{\text{graphs}} \longrightarrow S_{\text{matroids}}$ inducing a morphism of bialgebras

$G \longmapsto M(G)$ matroid of G

(we forget the number of connected components).

We get similar expressions for $X(S)$ and $T_S(M)$, and a universal convolution formula that specializes again to EL/KRS, but also to many others, for example Kung's formula:

$$T_M(1-ab, 1-cd) = \sum_{A \subseteq E(G)} a^{rk M - rk A} d^{|A| - rk A} T_{M|A}(1-a, 1-c)\, T_{M/A}(1-b, 1-d)$$

Example 3: $S = \{$arithmetic matroids$\}$ $S \ni a = (M, m)$

where M is a matroid and $m: 2^E \to \mathbb{N}$ satisfies some axioms.

Restriction is defined by $m_{(a|A)} = (m_a)|_A$, contraction by $m_{a/A}(B) = m_a(A \cup B)$, $\oplus$ by the product of multiplicities

$\Rightarrow S$ is a MMS. $\mathbb{K}S$ is not connected, as $m(\emptyset)$ can be any positive integer: $S_0 \simeq \{\mathbb{N} \setminus 0\}, \cdot \}$.

The universal Tutte character is

$$T_S(M, m) = \sum_{A \subseteq E} m(A)\, u_1^{\operatorname{rk} M|_A} v_1^{\operatorname{cork} M|_A} u_2^{\operatorname{rk} M/A} v_2^{\operatorname{cork} M/A}$$

which specializes to the arithmetic Tutte polynomial

$$T_{(M,m)}(x, y) = \sum_{A \subseteq E} m(A) (x-1)^{\operatorname{rk} E - \operatorname{rk} A} (y-1)^{|A| - \operatorname{rk} A}$$

The latter polynomial proved to have a wide number of applications, ranging from vector partition functions to polytopes to coloring of cell complexes....
In particular, it specializes to the Poincaré polynomial of toric arrangements, likewise the classical Tutte polynomial specializes to the Poincaré pol. of hyperplane arr.

Theor (Delucchi, M) If (M, m_1) and (M, m_2) are arithmetic matroids, then also $(M, m_1 \cdot m_2)$ is.

How do the corresponding arithmetic Tutte polynomials relate?

Let Ari be the MMS of arithmetic matroids, Mat be the MMS of matroids
we have morphisms $\mathrm{Mat} \to \mathrm{Ari}$ and $\mathrm{Ari} \to \mathrm{Mat}$

$M \mapsto (M, 1)$ $\quad$ $(M, m) \mapsto M$

Let us consider the MMS $\mathrm{Ari} \times_{\mathrm{Mat}} \mathrm{Ari}$ (whose objects are (M, m_1, m_2))
The universal convolution formula for this MMS specializes to a convolution formula for the arithmetic Tutte:

<u>Theor (DFM)</u> $T_{(M, m_1 \cdot m_2)} = T_{(M, m_1)} * T_{(M, m_2)}$

or more explicitly: $T_{(M, m_1 m_2)}(x, y) = \sum_{A \subseteq E} T_{(M, m_1)|_A}(0, y)\, T_{(M, m_2)/A}(x, 0)$.

This generalizes a formula by Backman and Lenz.
We also obtain more general results, such as an <u>arithmetic analogue of Kung's formula</u>.

OTHER APPLICATIONS

matroid perspectives → Las Vergnas poly.

delta-matroids → Bollobas-Riordan poly

delta-matroid perspectives → Krushkal poly

The target ring of the UTC is <u>not</u> polynomial!

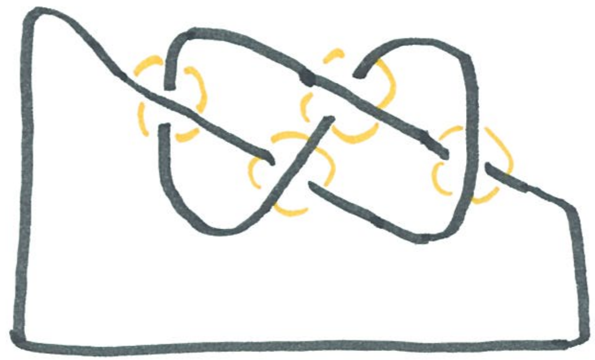
relative matroids → "relative Tutte poly"

Also in these cases we find new convolution formulae!

4. APPLICATIONS TO KNOTS

Def A **Knot** is an embedding of a circle S^1 into $\mathbb{R}^3$ (or S^3).
$L \subseteq \mathbb{R}^3$ is a **link** if every connected component of L is a knot.

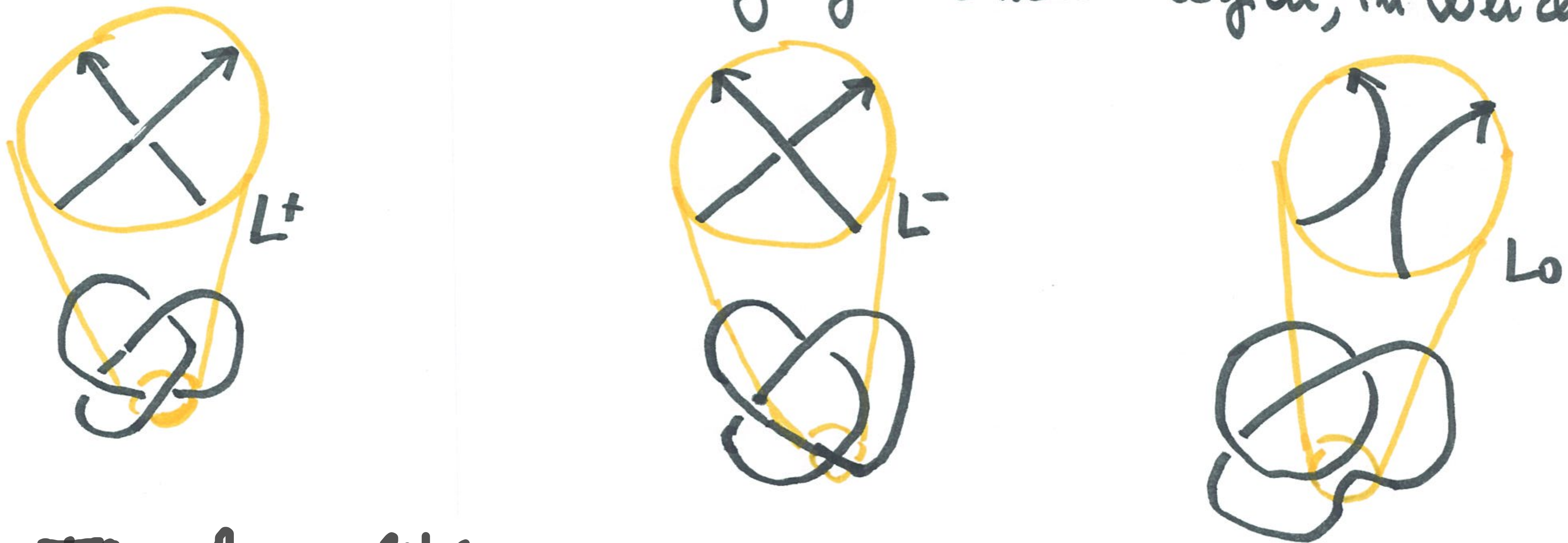
A link can be represented by a planar diagram, which is a (decorated) planar quadrivalent graph Q.



Def Two links L, L' are equivalent if there exist an orientation-preserving homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L) = L'$.

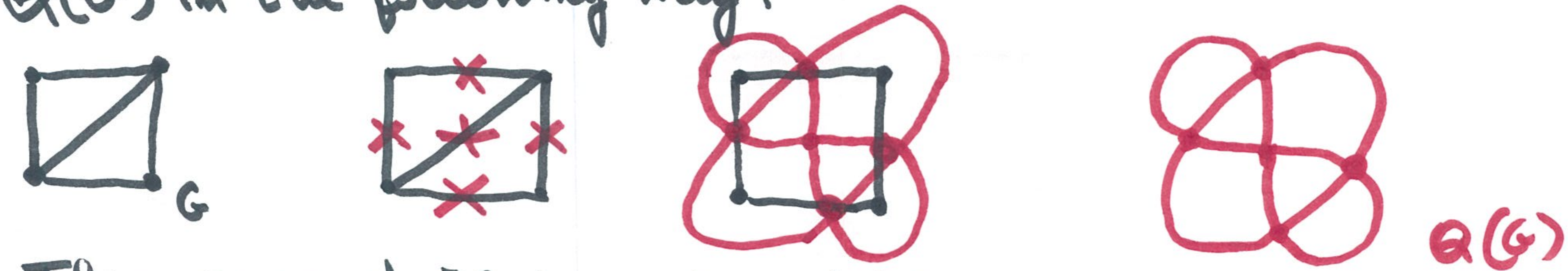
In general it is difficult to say if two links are equivalent: so, many invariants were introduced to distinguish them!

Def The Jones polynomial is the only $J(L) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ such that $J(\text{unknot}) = 1$ and $(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) J(L_0) = t^{-1} J(L_+) - t J(L_-)$ where L_0, L_+, L_- differ only by a small region, in which



Rem This looks like some sort of "deletion-contraction"...
How does this relate to graphs?

To a graph G we associate a quadrivalent planar graph $Q(G)$ in the following way:



This procedure is invertible:

Via this procedure, the diagram of a link corresponds to a graph with signed edges:

So we need a "Tutte polynomial for signed graphs", or more generally, for "colored matroids" (in our case two colors: $\{+,-\}$)

Example 4 $S = \mathrm{CMat}$ **colored matroids** $\mathrm{CMat} \ni (M, c)$ where M is a matroid and $c : E(M) \to C$ a map into a finite set C of "colors". (Remark: we can do the same with any MHS!).

Grothendieck monoid: a coloop and a loop for each color!

$\Rightarrow$ generated by $\{u_i = [\bullet \xrightarrow{i} \bullet],\ v_i = [\bullet\!\circlearrowleft_i],\ i \in C\}$ and free commutative on them. So

$$
\begin{aligned}
X(\mathrm{CMat}) &\hookrightarrow X(\mathrm{Mat}) \times \{a_i, i \in C\} \\
u_i &\longmapsto u\, a_i \\
v_i &\longmapsto v\, a_i \\
[M, c] &\longmapsto u^{\mathrm{rk}\, M}\, v^{\mathrm{cork}\, M} \prod_{e \in E(M)} a_{c(e)}
\end{aligned}
$$

The universal Tutte character specializes to the colored Tutte polynomial $T_{(M,c)}(x,y) = \sum_{A \in \mathcal{E}(M)} \left(\prod_{e \in A} a_{c(e)} \right) (x-1)^{\mathrm{rk}\,E - \mathrm{rk}\,A} (y-1)^{|A| - \mathrm{rk}\,A}$

Rem This is basically the same as Sokal's "multivariate Tutte polynomial" arising in mechanical statistics".

Theor (Kauffmann, §9) The Jones polynomial of a link L can be obtained as a specialization of the colored Tutte polynomial of the signed graph $Q^{-1}(L)$.

Our universal convolution formula implies thena some formulae for the Jones polynomiall. Work in progress: we don't know yet if those are new, and if they have relevant applications in knot theory...

THANK YOU!!!