Delta-matroids of dessins d'enfants and an action of the absolute Galois group Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)

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Motivation

Theorem (Whitney's planarity criterion)

A graph is planar if, and only if its graphic matroid is cographic.

Corollary

If G is a plane graph, then $\mathrm{M}^*(G) = \mathrm{M}(G^*)$, where G^* is the (geometric) dual of G.

Delta-matroids extend the previous corollary to any genus

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Definition

Definition (Matroids)

A non-empty collection $\mathcal{B}$ of subsets of a finite set E is the collection of bases of a matroid if, and only if $\mathcal{B}$ satisfies the *exchange axiom*:

For all $A, B \in \mathcal{B}$ and $a \in A \setminus B$ there exists $b \in B \setminus A$ such that $(A \setminus a) \cup b \in \mathcal{B}$.

Definition (Delta-matroids)

A non-empty collection $\mathcal{F}$ of subsets of a finite set E is the collection of **feasible sets** of a delta-matroid if, and only if $\mathcal{F}$ satisfies the ***symmetric*** *exchange axiom*:

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Introduced by Bouchet in 1987.

- Every matroid is a delta matroid.
- Unless a delta-matroid is a matroid, feasible sets are not of the same size.
- Circuit and rank function axioms exist.
- Deletion, contraction and duality are defined similarly:
  - if e is not a *loop* (i.e. not an element in no feasible set) of a delta-matroid $D = D(E, \mathcal{F})$, then

    $$\mathcal{F}_{D/e} = (\mathcal{F} \setminus e, \{F \setminus e : F \in \mathcal{F} \text{ and such that } e \in \mathcal{F}\})$$

  - if e is not an *isthmus/coloop* (i.e. not an element of every feasible set), then

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Example: graphic delta-matroids

Consider a connected graph G cellularly embedded on a closed, connected (orientable) surface X, i.e.

- Vertices are points on the surface
- Edges are arcs on the surface (not necessarily geodesic)
- The connected components of $X \setminus G$, called *faces*, are homeomorphic to open 2-cells

Such embeddings are also called maps on surfaces, or dessins d'enfants if one is interested in Galois theory (more precsely Grothendieck-Teichmüller theory and anabelian geometry).

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Definition

Let G be a map on a surface X with edge-set E. A *quasi-tree* of X is a subset $A \subseteq E$ such that A (as a ribbon graph) is spanning and has exactly one boundary component.

Theorem (Bouchet)

The collection of quasi-trees of X is a delta-matroid.

Example

If G is a plane map, then the collection of its spanning trees is a delta-matroid.

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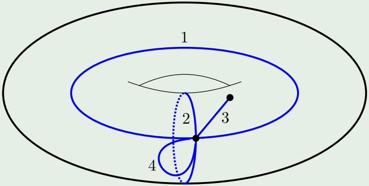

Feasible sets: 3 and 123

The topological genus of the surface of embedding is encoded by the *width* of the corresponding delta-matroid,

$$2g = \max\{|A| : A \in \mathcal{F}\} - \min\{|B| : B \in \mathcal{F}\}.$$

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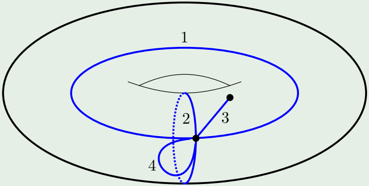

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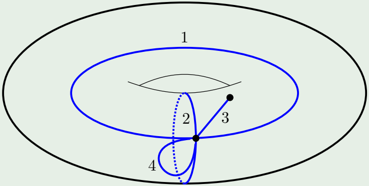

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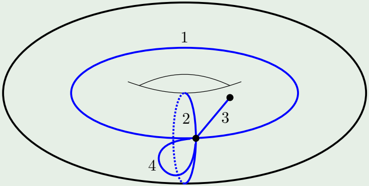

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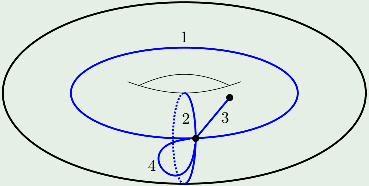

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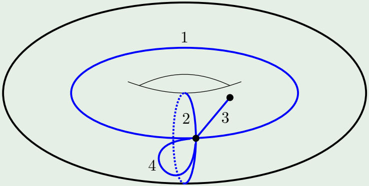

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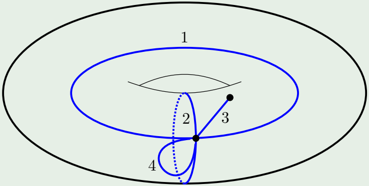

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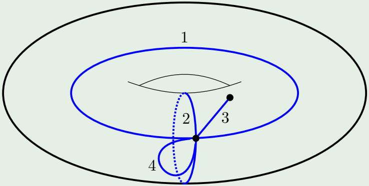

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Representability

Graphic delta-matroids are representable!

A map G on a surface X with edge-set $E = \{e_1, \ldots, e_n\}$ and dual-edge-set $E^* = \{e_1^*, \ldots, e_n^*\}$ is represented by an n-dimensional isotropic subspace of the $2n$-dimensional orthogonal vector space $\mathbb{Q}^E \oplus \mathbb{Q}^{E^*}$.

In general, the standard orthogonal $2n$-space has a basis

$$\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$$

and a symmetric bilinear form $(\cdot, \cdot)$ such that $(e_i, e_i^*) = (e_i^*, e_i) = 1$ for all e_i, and $(e_i, e_j) = 0$ for $e_i \neq e_j^*$.

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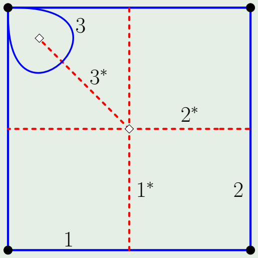

$$\begin{array}{cccccc} 1 & 2 & 3 & 1^* & 2^* & 3^* \end{array}$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Admissible 3-sets of column indices (i.e. those in which i and i^* don't appear together) such that the corresponding determinant is non-zero correspond to the feasible sets of G:

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The feasible sets are obained by intersecting with $E = \{1, 2, 3\}$.

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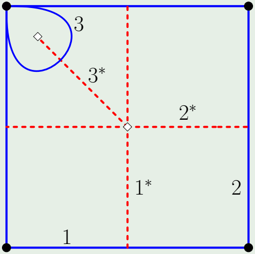

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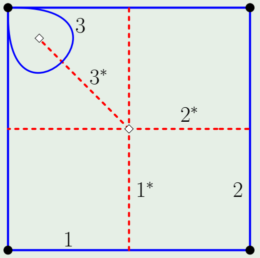

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Admissible 3-sets of column indices (i.e. those in which i and i^* don't appear together) such that the corresponding determinant is non-zero correspond to the feasible sets of G:

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The feasible sets are obained by intersecting with $E = \{1, 2, 3\}$.

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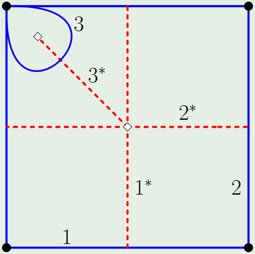

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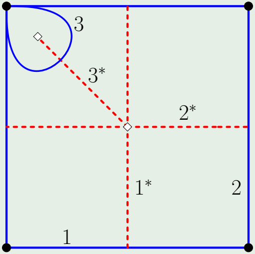

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Lagrangian matroids

The n-dimensional isotropic subspaces of a $2n$-dimensional symplectic or orthogonal space are called Lagrangian subspaces (n is the maximal dimension of an isotropic subspace).

Representable delta-matroids are represented by Lagrangian subspaces!

Definition

Let $E = \{e_1, \ldots, e_n\}$ and $E^* = \{e_1^*, \ldots, e_n^*\}$. A collection $\mathcal{B}$ of admissible n-subsets of $E \cup E^*$ is the set of **bases** of a *Lagrangian matroid* if, and only if

for all $A, B \in \mathcal{B}$ and $e, e^* \in A \triangle B$ there exist $f, f^* \in A \triangle B$ such that $A \triangle \{e, e^*, f, f^*\} \in \mathcal{B}$.

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Maps, revisited

Let G be a map on a surface X with edge set $E = \{e_1, \ldots, e_n\}$ and dual-edge set $E^* = \{e_1^*, \ldots, e_n^*\}$. A *base* of G is an admissible n-subset B of $E \cup E^*$ such that $X \setminus B$ is connected.

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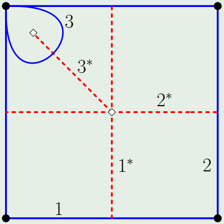

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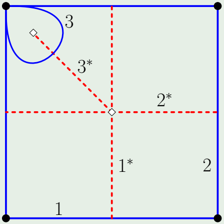

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General framework, Coxeter matroids

As with matroids, there are symplectic matroids that are not representable, i.e. that do not arise from isotropic subspaces of symplectic spaces. The general framework is provided by Coxeter groups and *Coxeter matroids* with respect to a finite reflection group W.

- When W is of type A_n, the resulting Coxeter matroid is a matroid.
- When W is of type BC_n, the resulting Coxeter matroid is a symplectic matroid.
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A dessin d'enfant is a connected bipartite graph embedded cellularly on a closed, connected and orientable surface X.

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Equivalently, a dessin d'enfant is a pair (X, f) of a compact Riemann surface X and a holomorphic ramified covering of the Riemann sphere $f : X \to \mathbb{CP}^1$, ramified at most over a subset of $\{0, 1, \infty\}$.

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A smooth projective algebraic curve X defined over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if, and only if X admits a holomorphic ramified covering of the Riemann sphere $f : X \to \mathbb{CP}^1$, ramified at most over a subset of $\{0, 1, \infty\}$.

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The absolute Galois group of $\mathbb{Q}$

Belyi's theorem tells us that any dessin d'enfant (X, f) is defined over $\overline{\mathbb{Q}}$, i.e. the coefficients of X and f are algebraic. Therefore, there is a natural action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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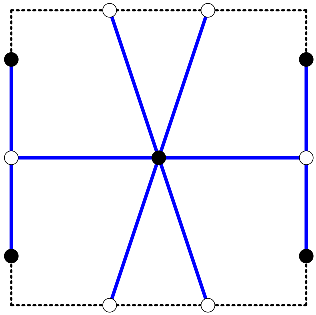

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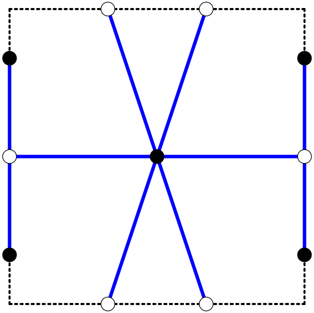

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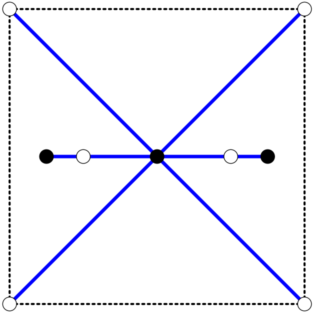

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Faithful action

This action is faithful, so in principle the elements of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) are completely encoded by the orbits of this action on dessins.

How to distinguish between the orbits? We look for invariants that can be computed (in a reasonable amount of time).

Some known invariants:

- The genus
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Question (A. Borovik)

Are Lagrangian matroids an invariant of the action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on the set of dessins?

Answer (GM, 2013-14)

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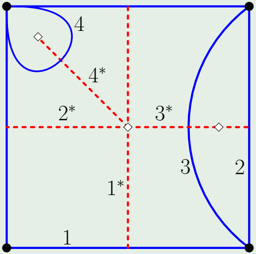

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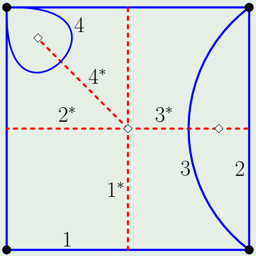

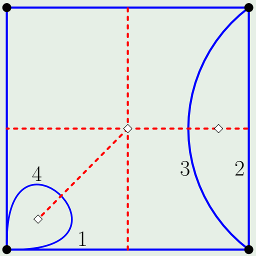

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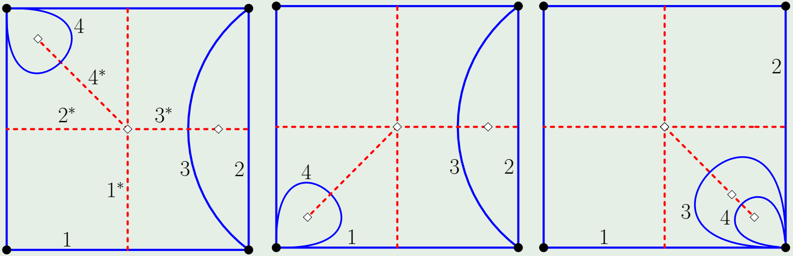

Thank you!

Merci Monsieur Chat!

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